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Short review on variational approach in abstract splines

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The abstract variational theory of splines in the Hilbert space originated from the well-known paper by M. Atteia (1965) and supported by P.-J. Laurent's researches (1968, 1973) is today a well-developed field in the approximation theory. We mean that the forthcoming researches in abstract theory were initiated by the problem of high-quality approximation of the functions at the multi-dimensional scattered meshes. But the efforts in this particular problem have already led to more powerful results both in abstract theory and in practice: new kinds of characterization theorems, convergence and general estimation techniques, finite element approach in the construction of complicated non-polynomial splines, theory of traces of splines onto smooth manifolds (new algorithms for the approximation of complicated surfaces in engineering), general theory of tensor splines, including the famous blending splines, variational theory of the vector and rational splines and so on. We think that the investigations of the Russian mathematicians from the Computing Center of USSR Academy of Sciences in Novosibirsk during recent years and after first fundamental successes of our French collegues, were quite significant in each of these fields. And the last but not least: the powerful software library based on these theoretical grounds was also created in Novosibirsk.

Indeed, this paper is only a short review and is not complete. We inform the reader who has interest in theory of splines and its various applications that the book *Bezhaev A.Yu., Vasilenko V.A. Variational Spline Theory* (255p.) will be published in special issue 3 (1993) of the Bulletin of Novosibirsk Computing Center, series Numerical Analysis, till summer 1993, see also [5].

1. Main definitions and properties [1, 3, 5]

Let X, Y, Z be the Hilbert spaces with the scalar products $(\ ,\ )_X$, $(\ ,\ )_Y$, $(\ ,\ )_Z$, $T : X \to Y$, $A : X \to Z$ be two linear bounded operators, and $z \in Z$ be fixed element.

Definition 1.1. We call $\sigma \in X$ an interpolating spline if

$$\sigma = \arg \min_{x \in A^{-1}(z)} \|Tx\|_Y^2, \qquad A^{-1}(z) = \{x \in X : \ Ax = z\} \neq \emptyset. \tag{1.1}$$

Definition 1.2. We call $\sigma_\alpha \in X$ a smoothing spline if

$$\sigma_\alpha = \arg\min_{x\in X} (\alpha\|Tx\|_Y^2 + \|Ax - z\|_Z^2), \quad \alpha > 0. \tag{1.2}$$

We denote by $N(T)$ and $N(A)$ the null spaces of the operators T and A, and by $R(T)$ and $R(A)$ their images. We suppose that $R(T) = Y$, $R(A) = Z$.

Theorem 1.1. *The interpolating and smoothing splines σ and σ_α exist and are unique for every $z \in Z$ if and only if $N(A) + N(T)$ is closed in the space X and $N(A) \cap N(T) = \{\theta_X\}$, θ_X is null vector of X.*

Remark. If $N(T)$ has the finite dimension (or codimension) or if $N(A)$ has the finite dimension (or codimension), then $N(A) + N(T)$ is always closed in X.

Theorem 1.2. *Following properties of orthogonaly for the interpolating and smoothing splines σ and σ_α take place*

$$\forall x \in A^{-1}(z) \quad \|T(\sigma - x)\|_Y^2 = \|Tx\|_Y^2 - \|T\sigma\|_Y^2, \tag{1.3}$$

$$\forall x \in X \qquad \alpha(Tx, T\sigma)_Y + (Ax - z, A\sigma - z)_Z = -(z, A\sigma - z)_Z. \tag{1.4}$$

Theorem 1.3. *The interpolating and smoothing splines σ and σ_α satisfy the following operator equations*

$$\begin{pmatrix} T^*T & A^* \\ A & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ \lambda \end{pmatrix} = \begin{pmatrix} \theta_X \\ z \end{pmatrix}, \tag{1.5}$$

$$(\alpha T^*T + A^*A)\sigma_\alpha = A^*z, \tag{1.6}$$

here T^*, A^* are adjoint operator with respect to T, A, and $\lambda \in Z$ is the Lagrangian parameter.

Definition 1.3. If $N(A) + N(T)$ is closed in X and $N(A) \cap N(T) = \{\theta_X\}$, then we say that (T, A) form a spline-pair.

Theorem 1.4. *If (T, A) is a spline-pair, then*

$$\|x\|_{(T,A)} = \left[\|Tx\|_Y^2 + \|Ax\|_Z^2\right]^{1/2} \tag{1.7}$$

is the norm which is equivalent to the norm $\|x\|_X$.

Definition 1.4. Spline pair (T, A) is submitted by the other spline-pair $(T, \tilde{A})$ if $N(\tilde{A}) \supset N(A)$, but $N(\tilde{A}) \neq N(A)$.

Definition 1.5. We say that spline-pair $(T, \tilde{A})$ is maximal with respect to spline-pair (T, A) if there is no other spline pair which submits it.

Maximal spline-pair does always exist, but is not unique.

Theorem 1.5. *If* $(T, \tilde{A})$ *is the maximal spline-pair then*

$$X = N(\tilde{A}) + N(T). \tag{1.8}$$

2. Characterization of abstract splines [19, 21, 5]

Let $\Omega \subset R^n$ be some domain and $X(\Omega)$ be the Hilbert space which is continuously embedded to the space $C(\Omega)$ of the continuous functions, i.e.,

$$\forall u \in X(\Omega) \quad \|u\|_{C(\Omega)} \leq C \cdot \|u\|_{X(\Omega)}. \tag{2.1}$$

It means that the point functional $k_t(u) = u(t)$ is continuous for any point $t \in \Omega$.

Definition 2.1. The function $G(s,t) : \Omega \times \Omega \to R^1$ is said to be the reproducing kernel of $X(\Omega)$ if for any point $t \in \Omega$ the function $g_t(s) = G(s,t)$ belongs to $X(\Omega)$ as a function of the variable s and

$$\forall t \in \Omega \; \forall u \in X(\Omega) \quad u(t) = (G(s,t), u(s))_{X(\Omega)}. \tag{2.2}$$

The reproducing kernel of the Hilbert space $X(\Omega)$ always exists and is unique, and $G(s,t) = G(t,s)$.

Let us consider in the Hilbert space $X(\Omega)$ the closed subspace P and connect with it the seminorm $|u|_P$ with the properties

$$\begin{aligned} &|u|_P = 0 \quad \text{iff } u \in P, \\ &|u|_P \leq C\|u\|_{X(\Omega)}, \end{aligned} \tag{2.3}$$

and with other usual properties of semiscalar product. This structure we denote by $(X(\Omega), |\cdot|_P)$.

Definition 2.2. The function $G_P(s,t)$ is said to be the reproducing kernel of the semi-Hilbert space $(X(\Omega), |\cdot|_P)$ if for any functional $L \in X^*$ the function $f(s) = LG(s,\cdot)$ lies in $X(\Omega)$ and any functional $L \in X^*$ vanishing on P can be represented by formula

$$L(u) = (LG_P(s,\cdot), u(s))_P \quad \forall u \in X(\Omega). \tag{2.4}$$

The reproducing kernel $G_P(s,t)$ also exists, symmetric, and may be is not unique.

Let Y, Z be two Hilbert spaces and $T : X(\Omega) \to Y$, $A : X(\Omega) \to Z$ be linear bounded operators, $z \in Z$ be fixed element. We suppose that (T, A) form the spline-pair, and $A^{-1}(z) \neq \emptyset$.

Theorem 2.1. *The interpolating spline σ as a solution of the variational problem*

$$\sigma = \arg \min_{x \in A^{-1}(z)} \|Tx\|_Y^2$$

can be always represented in the form

$$\sigma(t) = (\lambda, AG_P(\cdot, t))_Z + n(t), \tag{2.3}$$

where $t \in \Omega$, $\lambda \in Z$, $P = N(T)$, $n(t) \in P$.

If the reproducing kernel is known in the explicit form, then the interpolating spline σ can be found from the interpolation condition $A\sigma = z$.

3. General convergence theorems and error estimation techniques [3, 4, 5]

Let $T : X \to Y$ be the linear bounded operator with the closed image and finite dimensional null-space, $A_i : X \to Z_i$, $i = 1, 2, \ldots$ be also linear bounded operators to the Hilbert spaces Z_i, $i = 1, 2, \ldots$, and φ_* be the fixed element in X. We approximate $\varphi_* \in X$ by the interpolating splines σ_N, $N \geq N_0$. Each of σ_N is the solution of the problem

$$\begin{gathered} \sigma_N = \arg \min_{x \in M_N} \|Tx\|_Y^2, \\ M_N = \{x \in X : \; A_i x = A_i \varphi_*, \; i = 1, 2, \ldots, N\}. \end{gathered} \tag{3.1}$$

We suppose that the existence and uniqueness of σ_N take place when $N \geq N_0$. We denote by

$$\mathcal{A} = \{A_i : \; X \to Z_i\}_{i=1}^{\infty} \tag{3.2}$$

the infinite system of operators.

Definition 3.1. We say the sequence $x_n \in X$ tends to $x \in X$ by the system of operators $\mathcal{A}$ (symbolically $x_n \xrightarrow{\mathcal{A}} x$) if for every operator $A_i \in \mathcal{A}$ we have

$$\lim_{n \to \infty} \|A_i x_n - A_i x\|_{Z_i} = 0.$$

Definition 3.2. We call $\mathcal{A}$ the correct system of operators if the convergence $x_n \xrightarrow{\mathcal{A}} x$ implies the weak convergence of the sequence x_n to x on the set K which is dense in the space X,

$$\left[x_n \xrightarrow{\mathcal{A}} x\right] \Rightarrow \left[\exists K \subset X, \bar{K} = X,: \ \forall k \in K \ (k, x_n)_X \to (k, x)_X\right]. \tag{3.3}$$

Theorem 3.1. *The interpolating splines σ_N strongly converge to φ_* if and only if the system $\mathcal{A}$ is correct.*

If we can show the existence of the dense set K where the interpolating splines weakly converge to φ_*, then the strong convergence of σ_N to φ_* takes place.

Let B be any compact in R^n and the mapping $k : B \to X'$ be given, i.e., for every point $p \in B$ the linear bounded functional k_p is determined. Let us consider the interpolating problem

$$\begin{aligned} \sigma &= \arg \min_{x \in M_B} \|Tx\|_Y^2, \\ M_B &= \{x \in X : \ k_p(x) = k_p(\varphi_*), \ p \in B\}. \end{aligned} \tag{3.4}$$

We denote by B_ε some ε-net in the compact B and consider the interpolating problem

$$\begin{aligned} \sigma_\varepsilon &= \arg \min_{x \in M_{B_\varepsilon}} \|Tx\|_Y^2, \\ M_{B_\varepsilon} &= \{x \in X : \ k_p(x) = k_p(\varphi_*), \ p \in B_\varepsilon\}. \end{aligned} \tag{3.5}$$

Theorem 3.2. *If the mapping $p \to k_p$ is continuous, then*

$$\lim_{\varepsilon \to 0} \|\sigma_\varepsilon - \sigma\|_X = 0. \tag{3.6}$$

One of the most interesting applications of this theorem is the convergence of D^m-splines at the scattered ε-nets. Let $\Omega \subset R^n$ be some bounded domain, and $B \subset \Omega$ be its closed subdomain. The ε-nets B_ε are condensated in B. If we consider the D^m-prolongation $\sigma \in W_2^m(\Omega)$ of the function $\varphi_* \in W_2^m(B)$, $m > n/2$, which is founded from the problem

$$\begin{aligned} \sigma &= \arg \min_{u \in M_B} \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_\Omega (D^\alpha u)^2 \Omega, \\ M_B &= \{u \in W_2^m(\Omega) : \ u|_B = \varphi_*\}, \end{aligned} \tag{3.7}$$

then the D^m-splines σ_ε at the scattered ε-meshes B_ε founded from the problem

$$\sigma_\varepsilon = \arg \min_{u \in M_{B_\varepsilon}} \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_\Omega (D^\alpha u)^2 d\Omega, \qquad M_{B_\varepsilon} = \{u \in W_2^m(\Omega) : u|_{B_\varepsilon} = \varphi_*\}, \tag{3.8}$$

strongly converge to σ in $W_2^m(\Omega)$-norm for the finite or infinite ε-nets B_ε for $\varepsilon \to 0$. The situation $B = \Omega$ is only particular case of this theorem. Thus, it is possible in practice to solve the interpolation spline-problem in the domain Ω with the simple geometry (for example, in parallelepiped) instead of the domain B with the complicated geometry.

The following aim is the demonstration of the error estimation techniques for the T-splines at the scattered meshes. Let Ω be some bounded domain in R^n with the cone condition and $X(\Omega)$ be the Hilbert space which is continuously embedded into the space $C(\Omega)$ of the continuous functions with the uniform norm. Suppose that $T : X(\Omega) \to Y(\Omega)$ be linear bounded operator to the other Hilbert space $Y(\Omega)$ with the closed image and finite dimensional null-space. Denote by $n_1(P)$, $n_2(P)$, $\ldots$, $n_q(P)$ the basis of this null space.

Definition 3.3. We say that the totality of points $\bar{P} = \{P_1, P_2, \ldots, P_q\}$ from Ω forms L-solvable set (L-set) if the interpolating problem

$$n(P) \in N(T), \quad n(P_i) = r_i, \quad i = \overline{1,q},$$

is the uniquelly solvable for any r_i, $i = \overline{1,q}$.

For any L-solvable set $\bar{P}$ the generalized Lagrangian interpolation operator $\pi_{\bar{P}} : X(\Omega) \to N(T)$ is determined by the problem

$$\pi_{\bar{P}} u \in N(T), \quad (\pi_{\bar{P}} u)(P_i) = u(P_i), \quad i = \overline{1,q}.$$

It is easy to see that every L-set $\bar{P}$ has some non-trivial closed neighbourhood B, which also consists of L-sets.

Let the initial space $X(\Omega)$ be continuously embedded to the semi-Hilbert space $V(\Omega)$. Then the following inequality takes place

$$\|u - \pi_{\bar{P}} u\|_{V(\Omega)} \le C \|Tu\|_{Y(\Omega)} \tag{3.9}$$

with the constant $C > 0$ independent of $u \in X(\Omega)$ and $\bar{P} \in B$.

Let the function $u(x)$ be defined in the ball S_h of the small radius h and P_0 be its center. Then the function $\bar{u}(t) = u(P_0 + ht)$ is defined in the unit ball S_1 with the center at zero point. We suppose now the seminorm $\|\cdot\|_V$ and the norm $\|\cdot\|_Y$ have the following homogenious properties

$$\begin{aligned} \|u(P_0+th)\|_{V(S_1)} &= f_1(h)\cdot\|u\|_{V(S_h)}, & (3.10)\\ \|Tu(P_0+th)\|_{Y(S_1)} &= f_2(h)\cdot\|Tu\|_{Y(S_h)}, & (3.11)\end{aligned}$$

where $f_1(h)$, $f_2(h)$ are positive functions of $h>0$. Using (3.10), (3.11) we obtain the following error estimate for the generalized Lagrangian interpolation in the small ball S_h:

$$\|u-\pi_P u\|_{V(S_h)} \le Cf_2(h)/f_1(h)\cdot\|Tu\|_{Y(S_h)} \tag{3.12}$$

and it is impossible to improve this estimate in the whole class $X(\Omega)$.

Definition 3.4. We call the family of finite covers $\{B_i^{(h)}\}_{h\le h_0}$ of Ω consisting of small balls with the radius h special if the inequalities take place

$$\|u\|_{V(\Omega)} \le K_1\sum_i \|u\|_{V(B_i^{(h)})}, \tag{3.13}$$

$$\sum_i \|Tu\|_{Y(B_i^{(h)})} \le K_2\|Tu\|_{Y(\Omega)} \tag{3.14}$$

with the constants $K_1, K_2>0$ independent of u and $h\le h_0$.

Theorem 3.3. *If it is possible to construct the special cover of the domain Ω and the function $u\in X(\Omega)$ vanishes at the h-net ω_h, $h\le h_0$, then*

$$\|u\|_{V(\Omega)} \le C_1 f_2(h)/f_1(h)\ \|Tu\|_{Y(\Omega)} \tag{3.15}$$

with $C_1=$ const.

In application of T-splines σ_h which interpolate the function $\varphi_*\in X(\Omega)$ at the scattered mesh ω_h it leads us to the estimate

$$\|\sigma_h-\varphi_*\|_{V(\Omega)} \le C_1 f_2(h)/f_1(h)\|T(\sigma_h-\varphi_*)\|_{Y(\Omega)}. \tag{3.16}$$

The last term in (3.16) also tends to zero in accordance to Theorem 3.2.

It is always possible to construct the special covers for the domain with the cone condition which provides the prolongation for the typical functional space. Really, let $V(\Omega)=W_q^k(\Omega)$ and $X(\Omega)=W_2^m(\Omega)$, $m>n/2$, be embedded to it under inequalities $2\le q\le\infty$, $k-n/q\le m-n/2$, except the case $k=m-n/2$ & $p=\infty$. For the seminorm

$$\|D^k u\|_{L_q(\Omega)} = \left(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\int_\Omega (D^\alpha u)^q d\Omega\right)^{1/q} \tag{3.17}$$

we have $f_1(h) = h^{k-n/q}$, $f_2(h) = h^{m-n/2}$, and error estimate for D^m-splines is

$$\|D^k(\sigma_h - \varphi_*)\|_{L_q(\Omega)} \leq C_2 h^{m-n/2-k+n/q} \|D^m(\sigma_h - \varphi_*)\|_{L_2(\Omega)}. \quad (3.18)$$

The same techniques can be applied to the error estimation of the D^m-splines by local integrals (weighted or not) because these interpolation conditions provide the existence of h-net, where the difference $\sigma_h - \varphi_*$ vanishes.

4. Variational principle for tensor splines [22, 23, 5]

Let X and Y be two Banach functional spaces. Then the tensor product of X and Y is the closure of the linear space $X \otimes Y$ of the finite sums of the form

$$z(x,y) = \sum_{i=1}^{n} f_i(x) g_i(y), \quad f_i \in X, \quad g_i \in Y, \quad n \in N,$$

by some cross-norm [27]. Let α_p, $1 \leq p \leq \infty$, be the family of uniform p-nuclear cross-norms [27], and $X \otimes_p Y$ be the closure of $X \otimes Y$ by α_p-cross-norm. It is possible to define the tensor product of two linear bounded operators $A \otimes B$, $A : X \to U$, $B : Y \to V$, and $A \otimes_p B$ is the natural expansion of $A \otimes B$ to $X \otimes_p Y$ by continuation. It is principal fact that for the Hilbert spaces X and Y the cross-norm α_2 is the standard Hilbert norm produced by the scalar product in $X \otimes Y$.

Let X_i, Y_i, Z_i, $i = 1, 2$, be Hilbert spaces, and $T_i : X_i \to Y_i$, $A_i : X_i \to Y_i$, $i = 1, 2$, be linear bounded operators, and (T_i, A_i) be spline-pairs. It means that the interpolation and smoothing problems

$$\sigma_i = \arg \min_{x \in A_i^{-1}(z_i)} \|T_i x\|_{Y_i}^2, \quad i = 1, 2,$$

$$\sigma_i^\alpha = \arg \min_{x \in X_i} \alpha \|T_i x\|_{Y_i}^2 + \|A_i x - z_i\|_{Z_i}^2, \quad i = 1, 2,$$

have the unique solutions when $z_i = A_i f_i$, $f_i \in X_i$, $i = 1, 2$. We denote the corresponding spline-interpolation operator and spline-smoothing operator by $S_{A_i} : X_i \to X_i$, $S_{A_i}^\alpha : X_i \to X_i$.

Theorem 4.1. *Let the spline-pair (T_1, A_1) be submitted by the spline-pair $(T_1, \tilde{A}_1)$ and (T_2, A_2) be submitted by $(T_2, \tilde{A}_2)$. Then*

$$S_{A_1} \otimes_2 S_{A_2} f = \arg \min_{x \in (A_1 \otimes_2 A_2)^{-1}(z)} \|T_1 \otimes_2 T_2 x\|^2 + \|T_1 \otimes_2 \tilde{A}_2 x\|^2 + \|\tilde{A}_1 \otimes_2 T_2 x\|^2$$

where $z = A_1 \otimes_2 A_2 f$, *symbol* $\otimes_2$ *is connected with* α_2*-cross-norm.*

As example, for the interpolating bicubic spline on the rectangle Ω with the boundary Γ and rectangular grid, the simplest variational principle is the following

$$\sigma = \arg \min_{u \in (A_1 \otimes_2 A_2)^{-1}(z)} \int_\Omega u_{xxyy}^2 d\Omega + \int_\Gamma \left[\frac{\partial^2 u}{\partial \tau^2}\right]^2 d\Gamma.$$

Here τ is the tangent vector of Γ.

Theorem 4.2. *For the spline-smoothing tensor process the following variational principle takes place*

$$S_{A_1}^{\alpha_1} \otimes_2 S_{A_2}^{\alpha_2} f = \arg \min_{x \in X_1 \otimes_2 X_2} \alpha_1\alpha_2 \|T_1 \otimes_2 T_2 x\|^2 + \alpha_1 \|T_1 \otimes_2 A_2\|^2 + \alpha_2 \|A_1 \otimes_2 T_2 x\|^2 + \|A_1 \otimes_2 A_2 x - z\|^2,$$

where $z = A_1 \otimes_2 A_2 f$.

The following theorem is the universal tool for the error estimation of the interpolating tensor splines.

Theorem 4.3. *Let* B_i *be the Banach space,* $D_i : X_i \to B_i$ *be linear bounded operator, and* $i = 1, 2$. *Then for every* $f \in X_1 \otimes_2 X_2$ *the estimate takes place*

$$\begin{aligned} \|D_1 \otimes_2 D_2(f - S_{A_1} \otimes_2 S_{A_2} f)\| \le\ & g_1 g_2 \|T_1(I_1 - S_{A_1}) \otimes_2 T_2(I_2 - s_{A_2}) f\| \\ & + g_1 \|T_1(I_1 - S_{A_1}) \otimes_2 D_2 f\| \\ & + g_2 \|D_1 \otimes_2 T_2(I_2 - S_{A_2}) f\|, \end{aligned}$$

where

$$g_i = \sup_{x \in N(A_i)} \|D_i x\|_{B_i} / \|T_i x\|_{Y_i} < \infty, \quad i = 1, 2.$$

As example, for the interpolating D^{m_1,m_2}-splines at the Cartesian product of two scattered meshes in $\Omega_1 \subset R^{n_1}$ and $\Omega_2 \subset R^{n_2}$ the following error estimate is valid

$$\|D^{\alpha_1,\alpha_2}(f - \sigma)\|_{L_{p_1,p_2}(\Omega_1 \times \Omega_2)} = o\left(g_1(h_1) g_2(h_2) + g_1(h_1) + g_2(h_2)\right),$$

where

$$g_i(h_i) = O\left(h_i^{m_1 - |\alpha_i| - n_i/2 + n_i/p_i}\right),$$

and $2 \le p_i \le \infty$, $|\alpha_i| - n_i/p_i \le m_i - n_i/2$, except $p_i = \infty$ & $|\alpha_i| = m_i - n_i/2$, $i = 1, 2$.

It is trivial fact that the tensor spline-approximation from the computational point of view can be reduced to the "pseudo-one-dimensional" approximations like in the well-known case of bicubic splines at the grid.

5. Traces of splines onto manifolds [18, 5]

The problem of interpolation of the function whose values are given at the points situated on the manifold is quite practical and extremely interesting from the following point of view. Suppose, for example, the manifold is the usual $3D$-sphere. By any way we are able to construct the parametrization of the whole sphere or its parts (local maps). After that we have some $2D$-interpolation problem, and traditional methods of spline interpolation can be used. But in this case the spline depends on the parametrization way. It is not natural from the geometric positions because the sphere is independent of the parametrization as geometric object, and the corresponding approximation of the function over the sphere seems natural both with this property. The trace technology provides this property and let construct the smooth interpolations over canonical (sphere, torus, cylinder and so on) and complicated engineering surfaces.

Let $\Omega \subset R^n$ be a simply connected bounded domain whose boundary Γ is an infinite differentiable $(n-1)$-dimensional manifold, and $\omega_h \subset \Gamma$ be some set. Consider the interpolating D^m-spline for $\varphi_* \in W_2^m(\Omega)$, $m > n/2$,

$$\begin{aligned} \sigma_h &= \arg \min_{u \in A_h^{-1}(\varphi_*)} \|D^m u\|^2_{L_2(\Omega)}, \\ A_h^{-1}(\varphi_*) &= \{u \in W_2^m(\Omega) : \; u|_{\omega_h} = \varphi_*|_{\omega_h}\}. \end{aligned} \tag{5.1}$$

Definition 5.1. The restriction σ_h^Γ of D^m-spline σ_h onto the manifold Γ is said to be the trace of D^m-spline onto manifold.

If Γ is not algebraic surface (not zero level surface for some n-variable polynomial of the degree $m-1$) and ω_h forms L-solvable set, then D^m-spline σ_h (and also its trace onto Γ) exists and is unique. If Γ is algebraic, then σ_h may be not unique. But in both cases the following theorem is valid.

Theorem 5.1. *Let ω_h be h-net in Γ. Then for every function $\varphi_* \in W_2^{m-1/2}(\Gamma)$ and for the sufficiently small h the trace of spline $\sigma_h^\Gamma \in W_2^{m-1/2}(\Gamma)$ does exist and is unique.*

The error estimation technique for the traces looks like the case of the usual D^m-splines. The following asimptotic error estimate takes place

$$\forall \varphi_* \in W_2^m(\Gamma), \quad \forall s \in [0, m-1) \quad \|\sigma_h^\Gamma - \varphi_*\|_{W_2^s(\Gamma)} = o(h^{m-s-1/2}). \quad (5.2)$$

The finite element techniques, which we describe in the following point also can be applied for the construction of the traces.

6. Splines in the finite dimensional subspaces [3, 4, 5]

The characterization of theorems for the variational splines lead us to the following conclusion: to construct the spline in the analytical form we need to know exactly the reproducing mapping or kernel of the corresponding Hilbert or semi-Hilbert space. In the particular case of D^m-splines we need to know the Green function of the polyharmonic operator Δ^m in the multidimensional domain Ω; but this function is known in the analytical form only for one-dimensional case. If the reproducing kernel is even known, then the second difficulty is the numerical solution of the linear algebraic system with the dense matrix. And the third difficulty is the complicated representation formula for the spline. But the main preferences of the splines in practical applications were good approximation properties and simple piecewise polynomial representation!

All these reasons suggest us the following simple idea: instead of complicated analytical solution of the variational spline-problem we find simple approximation of the exact solution using the finite element method. In this case we obtain sparse linear algebraic systems and simple piecewise polynomial representation formulae.

Let X, Y, Z be the Hilbert spaces, $T : X \to Y$, $A : X \to Z$ be two linear bounded operators which form the spline-pair (T, A). Thus, the interpolating spline

$$\sigma = \arg \min_{x \in A^{-1}(z)} \|Tx\|_Y^2 \quad (6.1)$$

under the constraint $A^{-1}(z) \neq \emptyset$ exists and is unique.

Consider now the family of the finite dimensional subspaces $\{E_\tau\}_{\tau>0}$, τ in the space X to be a real positive parameter, and

$$E_\tau = \text{span}\, \{\omega_1^\tau, \omega_2^\tau, \ldots, \omega_{n(\tau)}^\tau\}. \quad (6.2)$$

Definition 6.1. We call $\sigma^\tau \in E_\tau$ the interpolating spline in the subspace E_τ if

$$\sigma^\tau = \arg \min_{x_*^\tau \in \mathcal{M}_\tau(z)} \|Tx_*^\tau\|_Y^2, \quad (6.3)$$

$$\mathcal{M}_\tau(z) = \{x_*^\tau \in E_\tau : \ \|Ax_*^\tau - z\|_Z^2 = \min_{x^\tau \in E_\tau} \|Ax^\tau - z\|_Z^2\}. \qquad (6.4)$$

Definition 6.2. We call $\sigma_\alpha^\tau \in E_\tau$ the smoothing spline in the subspace E_τ if

$$\sigma_\alpha^\tau = \arg \min_{x^\tau \in E_\tau} \alpha\|Tx^\tau\|_Y^2 + \|Ac^\tau - z\|_Z^2, \quad \alpha > 0. \qquad (6.5)$$

It is trivial fact that σ^τ and σ_α^τ exist and are also unique. They can be represented in the form

$$\sigma^\tau = \sum_{k=1}^{n(\tau)} \lambda_k \omega_k^\tau, \quad \sigma_\alpha^\tau = \sum_{k=1}^{n(\tau)} \lambda_k^{(\alpha)} \omega_k^\tau$$

and the coefficients λ_k, λ_k^α satisfy the following linear algebraic equations:

$$\begin{pmatrix} T & A \\ A & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad (\alpha T + A)\lambda_\alpha = f, \qquad (6.6)$$

where

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n(\tau)})^T,$$
$$\lambda_\alpha = (\lambda_1^{(\alpha)}, \lambda_2^{(\alpha)}, \ldots, \lambda_{n(\tau)}^{(\alpha)})^T,$$
$$f = ((z, A\omega_1^\tau)_Z, (z, A\omega_2^\tau)_Z, \ldots, (z, A\omega_{n(\tau)}^\tau)_Z)^T,$$

$q = (q_1, q_2, \ldots, q_{n(\tau)})^T$ is the vector of the Lagrangian coefficients, and $n(\tau) \times n(\tau)$-matrices T and A are composed of the elements $(T\omega_i, T\omega_j)_Y$ and $(A\omega_i, A\omega_j)_Z$.

Definition 6.3. We say that E_τ converges to X ($E_\tau \to X$) if for every $x \in X$ the sequence $x_{\tau_k} \in E_{\tau_k}$ does exist such that $\|x - x_{\tau_k}\| \to 0$ when $\tau_k \to 0$.

Definition 6.4. We say that E_τ weakly converges to X ($E_\tau \overset{w}{\to} X$) if for every $x \in X$ the sequence $x_{\tau_k} \in E_{\tau_k}$ does exist such that $x_{\tau_k} \overset{w}{\to} x$ when $\tau_k \to 0$.

Theorem 6.1. *If $E_\tau \to X$, then*

$$\|\sigma^\tau - \sigma\|_X \to 0, \quad \|\sigma_\alpha^\tau - \sigma_\alpha\|_X \to 0, \quad \tau \to 0.$$

Theorem 6.2. *Let the operator A have the finite dimensional image and $E_\tau \overset{w}{\to} X$. Then*

$$\|\sigma^\tau - \sigma\|_X \to 0, \quad \|\sigma_\alpha^\tau - \sigma_\alpha\|_X \to 0, \quad \tau \to 0.$$

The following basic question is: how to preserve the finete error estimates which we have already obtained for the analytical spline and also for their finite dimensional analogs (for splines in the subspaces)?

We suppose now that the interpolating operator $A_h : X \to Z_h$ depends on real parameter $h > 0$ which tends to zero (for example, the parameter of density for the scattered mesh), $\varphi_* \in X$ is fixed element and $z_h = A_h\varphi_*$. Let us consider the analytical interpolation spline problem of the form

$$\sigma_h = \arg \min_{x \in A_h^{-1}(A_h\varphi_*)} \|Tx\|_Y^2 \tag{6.7}$$

and denote by $S_h : X \to X$ its resolvent operator. It is obvious that S_h is the projector in X which maps the whole space X to the space of interpolating splines $Sp(h) = S_hX$. Let us have the seminorm $\|x\|_V$ in X under embedding condition

$$\forall x \in X \quad \|x\|_V \le C\|x\|_X.$$

We suppose that the error estimate takes place

$$\forall \varphi_* \in X \quad \|\varphi_* - S_h\varphi_*\|_V \le C_1 g_1(h) \cdot \|T\varphi_*\|_Y \tag{6.8}$$

with some function $g_1(h) \to 0$, $h \to 0$.

Furthermore let $\{E_\tau\}_{\tau>0}$ be the family of the finite dimensional subspaces in X and $B_\tau : X \to E_\tau$ be the projector onto E_τ. Suppose that the error estimate also takes place

$$\forall \varphi_* \in X \quad \|\varphi_* - B_\tau\varphi_*\|_V \le C_2 g_2(\tau) \cdot \|T\varphi_*\|_Y \tag{6.9}$$

with $g_2(\tau) \to 0$, $\tau \to 0$.

Denote by $B_{\tau(h)}$ the restriction of the operator B_τ to the space of interpolating splines $Sp(h)$, i.e., $B_{\tau(h)} : Sp(h) \to E_\tau$. The following property is fundamental to provide the "preservation" of the error estimate for the splines in the subspaces E_τ:

$$\theta\left(Sp(h), B_{\tau(h)}Sp(h)\right) \le \theta_0 < 1, \tag{6.10}$$

where $\theta(E^1, E^2)$ means the angle [26] between two subspaces E^1, E^2,

$$\theta(E^1, E^2) = \|P_1 - P_2\|,$$

P_1, P_2 are the orthoprojectors on E_1 and E_2 correspondently, and the constant θ_0 is independent of h.

Under the inequality (6.10) the following error estimates for the interpolating and smoothing splines in the subspaces $E_\tau(h)$ take place

$$\begin{aligned} \|\varphi_* - \sigma_h^{\tau(h)}\|_V &\le [K_1 g_1(h) + K_2 g_2(\tau(h))] \cdot \|T\varphi_*\|_Y, & (6.11)\\ \|\sigma_{\alpha,h} - \sigma_{\alpha,h}^{\tau(h)}\|_V &\le K_3 g_2(\tau(h)) \cdot \|T\varphi_*\|_Y, & (6.12)\\ \|\sigma_h - \sigma_h^{\tau(h)}\|_V &\le K_3 g_2(\tau(h)) \cdot \|T\varphi_*\|_Y & (6.13) \end{aligned}$$

with the constants K_1, K_2, K_3 independent of h.

The abstract theory of splines in the finite dimensional subspaces repeats in the case of multidimensional D^m-splines at the scattered meshes the well-known finite element approach. Since the analytical D^m-spline is not polynomial and complicated in practical application, the simplest steps of the numerical algorithm are the following: cover the interpolation point by some simple domain Ω (parallelepiped as example); construct the rectangular grid in this domain and connect with it the space of polynomial finite elements with the suitable smoothness (multidimensional B-splines as example); assemble the sparse matrices $\boldsymbol{T}$ and $\boldsymbol{A}$ by the formulae

$$\begin{aligned} t_{ij} &= \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_\Omega (D^\alpha \omega_i \cdot D^\alpha \omega_j) d\Omega, \\ a_{ij} &= \sum_{P\in\omega_h} \Omega_i(P) \cdot \Omega_j(P), \end{aligned}$$

compute the right hand part $\boldsymbol{f}$ by formula

$$f_i = \sum_{P\in\omega_h} \varphi_*(P) \cdot \omega_i(P)$$

and solve the system (6.6) by iterative or direct method.

This finite element technique permits us to construct the universal software routines, which are suitable not only for the solution of the interpolation and smoothing spline problems at the multidimensional scattered meshes but also for the approximation by spline traces onto manifolds, by the tensor splines, rational splines [24], discontinuous splines [25], and for many other practically useful constructions. This finite element techniques is basic, especially for the various approximations at the multi-dimensional scattered meshes, in the software library LIDA-3 on data approximation (for example, see Appendix 2 in [5]).

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