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Quasi-stationary distribution function of the rotating collisionless gravitating disk*

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Abstract. We have constructed a quasi-stationary distribution function of the rotating collisionless gravitating disk with an oblate spheroid-like structure at the center. The construction technique is based on the numerical tracking of the evolution of the initially non-stationary thin disk on time scales of dozens of rotations. We show that a distribution function can be approximated using the axially symmetric density $\sigma(r, z)$ and the velocity dispersions $c_r(r)$, $c_\phi(r)$, $c_z(r)$. The numerical analysis of the disk stability in the vertical direction has been made. It was found that a forced compression of the disk height implies the development of a bending instability.

1. Introduction

For the study of the dynamics of collisionless gravitating systems (such as galaxies or a dust component in circumstellar disks) it is important to know parameters of the stationary distribution functions (DF) of the matter $f = f(\boldsymbol{x}, \boldsymbol{v})$ in a self-consistent gravitating field. First, it is useful in order to estimate an impact of the external forces produced by a central body, a gaseous component, a dark matter, etc. Second, applying perturbations of the known form to a stationary system it is possible to analyze their influence on the global stability of a DF, separating true physical instabilities from the numerical fluctuations and noises [1]. At last, while studying processes that are specific to the multiphase systems (e.g. a gas-dust circumstellar disk) it is necessary to make sure that non-stationary state of a component does not have a dramatic impact on the evolution of the whole system.

The fact is, the analytical reconstruction of the consistent pair — DF and gravitational field — is a very difficult task, and there are only a few analytical solutions for the disk systems [2, 3], most of them hardly corresponding to real systems, because they are unstable (for example, a uniformly rotating Maclaurin disk is one of those). Although analytical approaches help to do some qualitative estimations for the disk galaxies [4,5], at the same time they have restricted application for constructing the three-dimensional stationary rotating disk systems. The only way is to numerically solve equations of the stellar dynamics consisting of the collisionless Boltzmann equation (CBE) for

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DF (also known as the Vlasov equation in plasma physics) and the Poisson equation for the gravitational field:

$$\begin{aligned}
&\frac{\partial f}{\partial t} + \boldsymbol{u}\frac{\partial f}{\partial \boldsymbol{r}} - \nabla\Phi(t,\boldsymbol{r})\frac{\partial f}{\partial \boldsymbol{u}} = 0, \qquad f(0,\boldsymbol{r},\boldsymbol{u}) = f^0(\boldsymbol{r},\boldsymbol{u}),\\
&\Delta\Phi(t,\boldsymbol{r}) = 4\pi G\rho(t,\boldsymbol{r}), \qquad \Phi(t,\boldsymbol{r})|_{\boldsymbol{r}\to\infty} = 0, \\
&\rho(t,\boldsymbol{r}) = \int f(t,\boldsymbol{r},\boldsymbol{u})\,d\boldsymbol{u}.
\end{aligned} \tag{1}$$

The main idea of this approach is based on the numerical technique of tracking the disk evolution [6]. At the initial moment, $f^0(\boldsymbol{r},\boldsymbol{u})$ is given and assumed to be close to equilibrium state. Then system 1 is numerically integrated to the moment when it becomes clear either f is stable or not. Of course, it cannot be treated as a rigorous proof even in a case of the perfect stability of f in the course of simulation because of the two main reasons. The first one is possible artefact of a numerical code, which prevents the growth of instability. The second one is that some instabilities may have a growing time that exceeds the chosen time of numerical calculation. However for most applications it is sufficient to make sure that DF does not change during a characteristic time scale (e.g. several rotation periods for rotating disks). Henceforth, if DF is stationary in numerical simulation, it is called *quasistationary* to distinguish it from an exact stationary solution of the CBE-Poisson system.

The first attempts to numerically analyze stability of DF [7–9] have shown that this is a very powerful tool, but nevertheless it is of great computational costs. So, the later efforts were mainly concentrated on constructing DFs, which are close to the observable galaxies and include some external fields of the galactic halo, buldge and dark matter ([10–12] and references therein). But today, with the help of advanced numerical techniques [13–15] and with the use of supercomputers it becomes possible to solve system (1) on a large time scale without significant simplifications.

This paper presents results of the numerical simulations aimed at constructing quasistationary DF for the rotating collisionless disk. It is shown that a thin non-stationary disk evolves to the axially-symmetric quasistationary DF, which has a shape of a disk with a spheroid-like structure at the center. We briefly discuss some features of its evolution and then numerically test the stability of the obtained DF with regard to the growth of the bending instabilities.

2. A numerical model and basic tests

This section briefly describes a numerical model and is mainly concentrated on the test scenarios, which demonstrate its reliability and robustness in the case when a solution is unstable. A detailed description of the used numerical methods and parallelization techniques can be found in [15, 16].

To solve (1), we use a cylindrical system of coordinates. A CBE is solved using the PM (particle-mesh) method [13] (that is known as, particle-in-cell method in plasma physics [17]). One of the main features of the developed numerical code is that in order to find a gravitational field, we solve the Dirichlet problem for the Poisson equation, while some other implementations [18, 19] employ a convolution method (an extensive review of the methods of the gravitational field calculation can be found in [20]).

In addition to a usual analysis of the numerical convergence of the method, testing its computer implementations and checking the runtime accuracy criteria such as conservation laws, we have to make sure that:

- the growth of gravitational instabilities is not suppressed in the numerical integration of (1) (if these instabilities are present in the initial system of PDE).
- numerical instabilities, caused by interactions of particles with a mesh [17], do not appear for a stationary solution of (1).

If these requirements are met, then it allows us to conclude:

- if $f^0(\boldsymbol{r}, \boldsymbol{u})$ is stationary, then it remains stationary in the numerical integration of (1).
- if a numerical approximation of DF is quasi-stationary, then the corresponding solution of (1) is quasi-stationary as well.

To test these properties, it is useful to adapt well-known analytical solutions [2].

Instability of the uniformly rotating the Maclaurin disk. The Maclaurin disk is a thin disk (the vertical height is zero) with the surface density:

$$\sigma(r) = \begin{cases} \sigma_0 \sqrt{1 - \left(\frac{r}{R_0}\right)^2}, & r \le R_0, \\ 0, & r > R_0, \end{cases} \tag{2}$$

where R_0 is a radius of the disk and σ_0 is calculated from the condition

$$M_0 = 2\pi \int_0^{R_0} \sigma r \, dr = \frac{2\pi}{3} \sigma_0 R_0^2,$$

where M_0 is the mass of the disk.

The radial and the vertical velocities are $v_r(r) = 0$ and $v_z(r) = 0$, the azimuthal velocity $v_\phi(r)$ is derived from the equality of centrifugal and gravitational forces:

$$\Omega^2 r = -\frac{\partial \Phi}{\partial r}, \tag{3}$$

where Ω is an angular velocity (which is constant for the Maclaurin disk). It is known that the Maclaurin disk is in equilibrium, but unstable to the growth of the Jeans instabilities [2].

The first test is to make sure that DF, defined from (2),(3) and numerically implemented in the code, is in the equilibrium state. To perform this test, the Poisson equation is solved only at the initial moment and then a mesh function of the gravitational field is stored in the course of the simulation. So, the disk evolves in a constant field and indeed it has shown a perfect stability.

The second test is to check the Toomre criterion [21] of the stability of rotating disks against axially symmetric perturbations:

$$c_r \geq c_T = 3.36 \frac{G\sigma}{\kappa}, \tag{4}$$

where $\kappa = 2\Omega\sqrt{1 + \frac{r}{2\Omega}\frac{d\Omega}{dr}}$ is the epicyclic frequency. The Toomre parameter $Q = \frac{c_r}{c_T}$ is often used.

To implement this test, the gravitational field is calculated at each time step, but the azimuthal forces are neglected. This implies that all motions of the particles are axially symmetric. The results of numerical simulations are shown in Figure 1, for $Q = 0.1 \div 1.5$. Axially symmetric instabilities (rings) grow, when $Q < 1$ and are suppressed, when $Q > 1$, that corresponds to the analytical predictions.

The next test is to simulate a cold Maclaurin disk ($Q \ll 1$) in a self-consistent field. Since the disk is unstable, then it should fall into pieces,

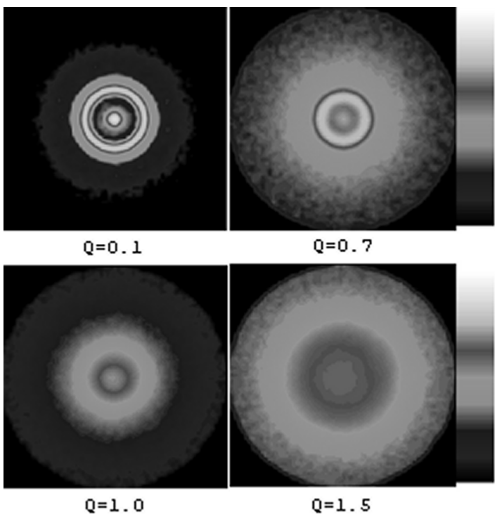

Figure 1. The surface density in the equatorial plane of the Maclaurin disk in axially symmetric field with different values of the Toomre Q parameter. Logarithmic density scale is at the right side

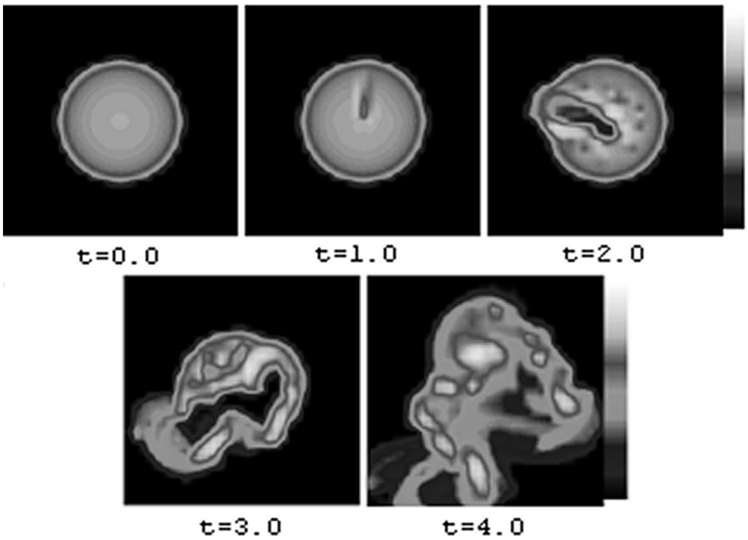

Figure 2. The surface density in the equatorial plane of the cold Maclaurin disk in a self-consistent field

which correspond to the Jeans masses. Figure 2 shows the fragmentation of the disk after less than 1/4 of its rotation period.

The last test is the self-consistent evolution of a hot Maclaurin disk ($Q = 1$) with a non-zero vertical height. In this case, the azimuthal dispersion was chosen from the Lindblad relation [5]:

$$c_\phi = c_r \frac{\kappa}{2\Omega}. \tag{5}$$

The results of simulation are presented in Figure 3. The disk is stable against the axially symmetric perturbations but unstable against the nonlinear non-axially symmetric bar instability. The results are in agreement with earlier works [7, 8] performed with different numerical techniques.

Stability of the Einstein model. The DF of the Einstein model for spherical galaxies or globular clusters of stars is defined as follows. Particles are distributed on a sphere (for simplicity, we take a uniform density). Radial velocities of particles in spheroidal coordinates are equal to zero. Tangent velocities are chosen with arbitrary directions and an absolute value corresponding to the rotation of the particle around the sphere's center. Since this DF is stable [2], then we expect the stability of the corresponding numerical approximation of the DF. Figure 4 shows the surface density of the sphere at $t = 0.0$ and $t = 10.5$: indeed, the sphere is stable.

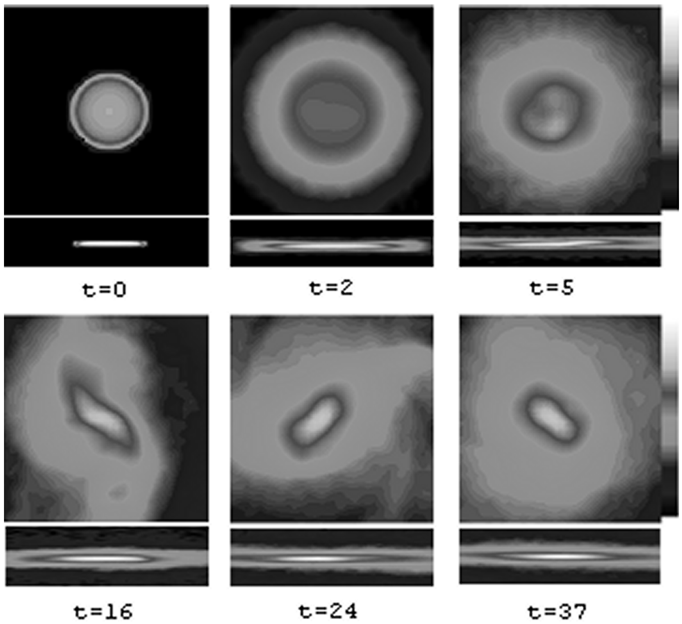

Figure 3. The surface density in the equatorial and in the meridian planes of the hot Maclaurin disk in a self-consistent field

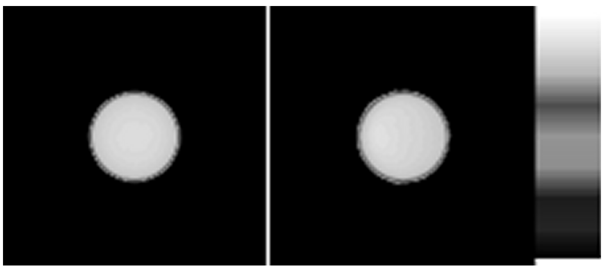

Figure 4. The surface density of the Einstein model in the equatorial plane

3. Evolution of a non-stationary thin disk

The surface density and the vertical density of the initial DF are given by the equations:

$$\sigma(r) = \begin{cases} \sigma_0 e^{-r/L}, & r \le R_0, \\ 0, & r > R_0, \end{cases}, \qquad \sigma(z) \propto \cosh^{-2}\Big(\frac{z}{z_0}\Big), \tag{6}$$

where L and z_0 are the density scale parameters. Equations (6) are considered to be the first simple approximation to the density of real disk galaxies [3, 5]. The mean velocities correspond to circular rotation (3) and initial dispersions are given by $c_r = 0.5$, $c_\phi = 0.25$, $c_z = 0.0$. These values of radial

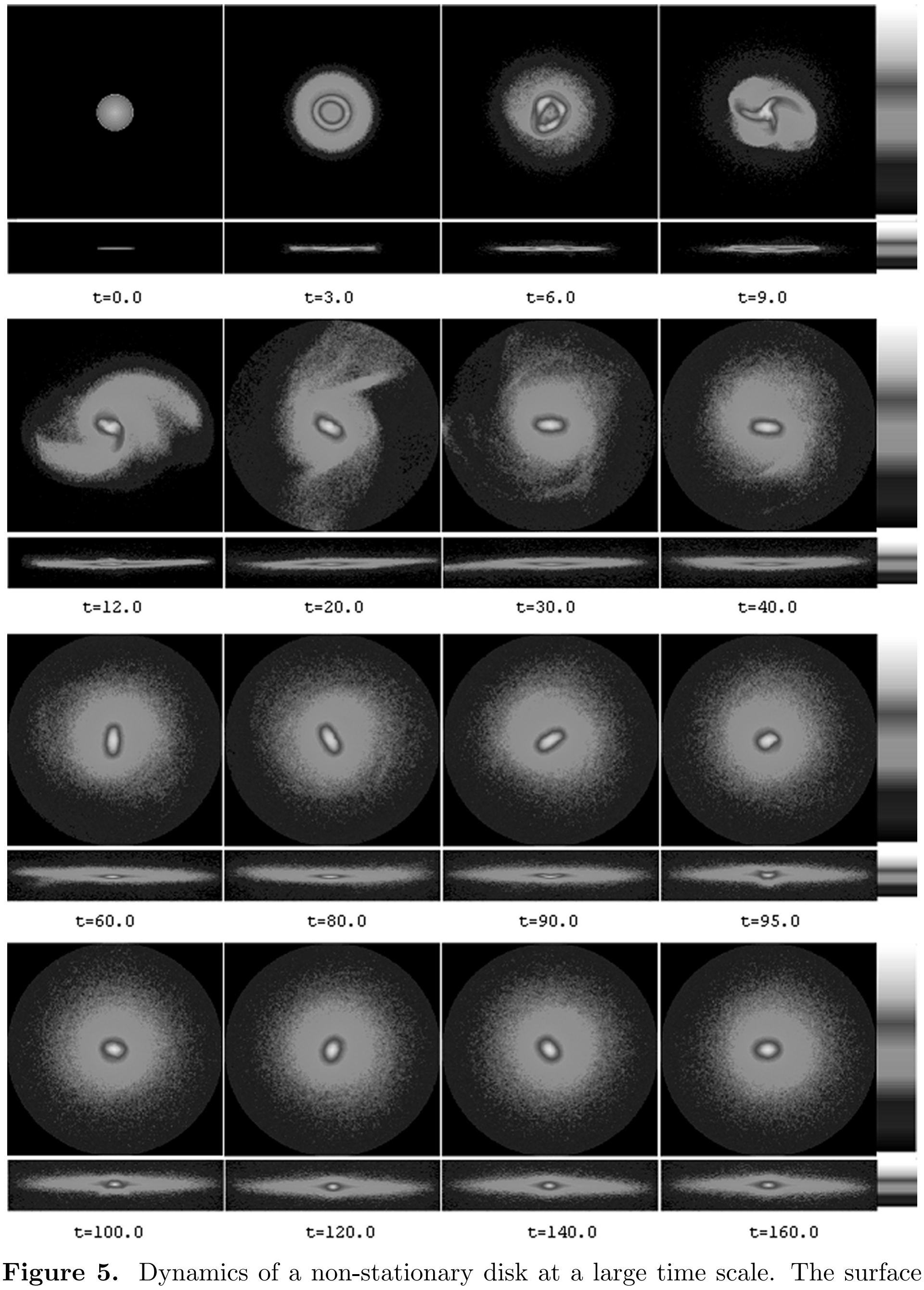

Figure 5. Dynamics of a non-stationary disk at a large time scale. The surface density in the equatorial and in the meridian planes of the disk for different time instants

and azimuthal dispersions do not satisfy equation (5), hence the initial DF is far from the equilibrium state.

Figure 5 shows the evolution of the DF. At the first moment ($t = 3.0$), we can see a ring-like structure produced by violation of the Lindblad relation (5). The density in the ring is stronger than in the initial disk, and the Toomre parameter Q becomes less than 1. This means that the Jeans instabilities are not suppressed, and the ring falls into pieces ($t = 6.0$), which collapse with the formation of a transient 3-arm spiral structure ($t = 9.0$). In less than half a rotation period, the spiral arms disappear, and we observe a rotating bar in the central area ($t = 12.0$). At the same time we can see bending deformations of the disk in the vertical direction ($t = 3.0$).

A subsequent bar evolution is the following. Beginning with $t = 20.0$ to $t = 90.0$ (six rotation periods) no significant changes are observed: the bar rotates with a weak slowing down of the angular velocity and demonstrates the stability both in the equatorial plane and in the vertical direction. But at the moment $t = 90.0$, one can notice the growth of the bending instability. It weakens the bar at the moment $t = 100.0$ forming an ellipsoid-like structure, which rotates with a constant angular velocity till the end of simulation ($t = 120.0$).

Results of this simulation: the growth of the bar instability, the bar weakening due to the growth of the secular bending instability are in good agreement with a number of publications ([22, 23] and references therein). The authors [24] state that they failed to find parameters of the initial DF when the bending instability in a bar leads to the formation of a bulge (an axially-symmetric bar). We have observed the same behavior: bending of a bar decreased the relation of the axis parameters, but could not make it completely axially-symmetric.

4. Construction of an axially-symmetric quasi-stationary disk

In order to construct an axially-symmetric DF, we take the DF obtained by the simulation in from the previous section and then take mean azimuthal values of the following functions: the density $\sigma(r, z)$ and the dispersions $c_r(r), c_\phi(r), c_z(r)$. It turns out that the DF approximated by these functions is close to the quasi-stationary state. Figure 6, shows the simulation results with approximated axially-symmetric DF at a time scale of four rotations of particles with the radial coordinates $r = 1$. It is clear that there are no significant modifications of DF. These results are in agreement with those presented in [8] (2D numerical model).

To validate the stability of the disk against the bending instabilities and to find if it is possible to construct a thin stable disk, we use the following method.

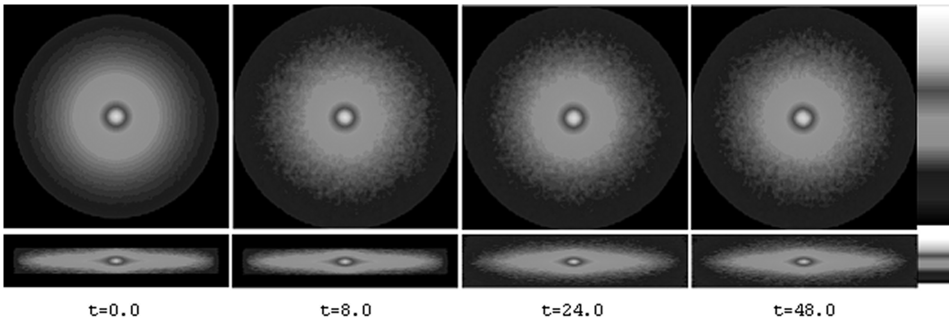

Figure 6. Dynamics of the axially-symmetric DF. The surface densities in the equatorial and in the meridian planes of the disk for different time instants

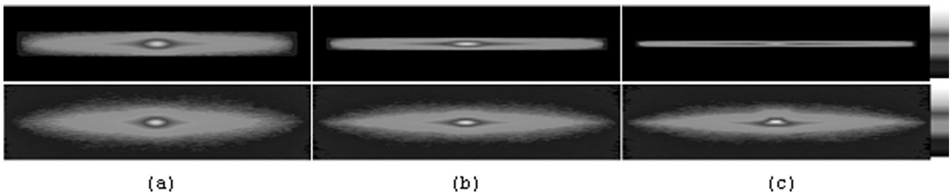

Figure 7. Dynamics of the quasi-stationary disks with different compression factors (a) $k_h = 1.0$, (b) $k_h = 0.5$, (c) $k_h = 0.25$. The upper row corresponds to the time $t = 0.0$, the lower row to the time $t = 22.5$. The surface density in the meridian plane of the disk

A disk is artificially compressed in the vertical direction with the factor k_h so that the vertical length is $z_h = k_h z_h^0$ (where z_h^0 is a vertical length of the disk). It was found that when k_h is less than $1/4$, there was a bending instability of a spheroid-like structure at the center. Figure 7 demonstrates the comparison of the results of the three simulations with the compression factors $k_h = 1.0,\ 0.5,\ 0.25$.

There are no significant differences with a compression factor $k_h = 0.5$, i.e., the disk is stable at a time scale of two rotation periods. However, with further compression to $k_h = 0.25$, one can observe the bending instability. Numerical simulation with $k_h = 0.25$ has been checked on a fine grid with a number of nodes in the vertical direction $M_z = 512$ and shows the same result. It is in agreement with simulations performed in [22, 25] and known observational data of the absence of elliptical galaxies with a compression factor exceeding E7.

5. Conclusion

Using the numerical model developed in [15,16] for solving problems of gravitational collisionless physics, we succeeded to construct quasi-stationary distribution function for the rotating three-dimensional collisionless gravitating axially-symmetric disk. We found that the obtained DF can be approximated using the density function $\sigma(r, z)$ and the velocity dispersions $c_r(r)$, $c_\phi(r)$, $c_z(r)$. Numerical simulations with a compressed DF in the vertical direction shows that it becomes unstable to the growth of bending instabilities if a disk is thin.

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