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On decomposability in logical calculi*

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Abstract. In the paper, a natural class of logical calculi is fixed for which we formulate the notion of a Δ–decomposable set of formulas. We demonstrate that the property of uniqueness of signature decompositions holds in those calculi of this class that have the Craig interpolation property. In conclusion, we give a sufficient condition for the Δ–decomposability property to be decidable.

1. Introduction

In Computer Science, decomposition is a standard technique to reduce complexity of problems. In Logic, the notion of decomposition appears in numerous applications including the important field of reasoning over theories. The main idea is to identify those fragments of a theory that are necessary and sufficient for testing a given property, thus reducing the search space and complexity of reasoning. There is a number of papers, in which decomposition methods for logical theories are considered. We can mention the syntactic graph-based approach to partitioning of theories [1] proposed in the scope of new methods for automated theorem proving. The entailment-based approaches to finding "independent parts" of theories in modal and description logics [10, 2, 3] and the semantic approaches [9] closely related to results on conservative extensions [7, 11] are extensively applied in the study of terminological systems.

In presence of interpolation, a natural approach is to consider signature partitions of theories. If a theory $\mathcal{T}$ is decomposed into a union of "self-contained" signature-disjoint theories, then, given a formula φ in signature σ, the entailment of φ by $\mathcal{T}$ reduces to the entailment of φ by only those decomposition components whose signatures have a non-empty intersection with σ. Which logical calculi allow for an algorithm to find signature decompositions for an arbitrary given set of formulas and when can this algorithm be chosen as deterministic? What properties should a calculus satisfy for every its set of formulas to have a unique signature decomposition? In this paper, we give a partial answer to these questions. We formalize the concept of signature decomposition via the general notion of Δ–decomposability and study the property of uniqueness of signature decompositions. This property guarantees that every theory has exactly one possible representation as

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a union of indecomposable theories. By specifying properties of the consequence relation, we define a class of calculi satisfying the property of uniqueness of signature decompositions and a class of calculi in which the problem of finding non-trivial decompositions for an arbitrary given set of formulas is decidable. Unsurprisingly, the considered notions are closely related to important interpolation properties studied in logics: the Craig interpolation property and uniform interpolation property. We try to formulate the ideas related to signature decomposability in the most general form, that is why the complexity issues as well as specific results proved for concrete calculi are intentionally left out of the scope of the paper. The results in this paper are directly transferred to the classical, intuitionistic logic, and a wide range of modal logics.

2. Preliminaries

This work is a generalization of results in [13]. In that paper, the notion of decomposable first-order theory was introduced and it was proved that each first-order theory has a unique canonical decomposition. As only syntactic properties of the first-order language were used in the proof, it appeared possible to extend it to a rather wide class of logical calculi. Thus, the proofs in this paper follow the ideas from [13] and extend the previous results from the point of view of studying a more general Δ–decomposability notion in a wide natural class of logical calculi.

All the claims in the paper are formulated with respect to a logical calculus $\mathcal{L}$ with the consequence relation $\vdash_{\mathcal{L}}$ satisfying properties defined below. We distinguish two disjoint subsets of symbols in the language of $\mathcal{L}$ — a set of logical and a set of non-logical symbols. For a formula φ in the language of $\mathcal{L}$, the signature of φ is the set of non-logical symbols that occur in φ. We assume that the length of each formula of $\mathcal{L}$ is finite. For two sets of formulas Γ and Λ in $\mathcal{L}$, the expression $\Gamma \vdash_{\mathcal{L}} \Lambda$ means that $\Gamma \vdash_{\mathcal{L}} \varphi$ for every formula $\varphi \in \Lambda$ (we say that the set Λ is entailed by Γ). The sets Γ and Λ are equivalent (abbrev. $\Gamma \sim_{\mathcal{L}} \Lambda$), if $\Gamma \vdash_{\mathcal{L}} \Lambda$ and $\Lambda \vdash_{\mathcal{L}} \Gamma$. The notation $\Gamma, \Lambda \vdash_{\mathcal{L}} \varphi$ traditionally means that $\Gamma \cup \Lambda \vdash_{\mathcal{L}} \varphi$.

We assume that the relation $\vdash_{\mathcal{L}}$ satisfies the extensionality, transitivity, compactness, adjunction, and tautology property. More precisely, for any formula φ of $\mathcal{L}$ and any two sets Γ and Λ of formulas in $\mathcal{L}$, the following conditions are satisfied:

- if $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathcal{L}} \varphi$ (ext.);
- $\Gamma \vdash_{\mathcal{L}} \Lambda$ and $\Lambda \vdash_{\mathcal{L}} \varphi$, then $\Gamma \vdash_{\mathcal{L}} \varphi$ (trans.);
- if $\Gamma \vdash_{\mathcal{L}} \varphi$, then there exists a finite $\Lambda \subseteq \Gamma$ such that $\Lambda \vdash_{\mathcal{L}} \varphi$ (comp.);

- for every finite set Γ of formulas in signature Σ, there exists a formula $\phi \in \mathcal{L}$ in signature Σ such that for every formula $\psi \in \mathcal{L}$, we have $\Gamma \vdash_{\mathcal{L}} \psi$ iff $\phi \vdash_{\mathcal{L}} \psi$ (adj.);
- for each finite set Σ of signature symbols, there exists a formula φ such that every symbol of Σ occurs in φ and $\varnothing \vdash_{\mathcal{L}} \varphi$ (taut.).

We suppose that for some logical symbol denoted here as $\bowtie$, for every set of formulas Γ and formulas φ and ψ, the calculus $\mathcal{L}$ satisfies the deduction property in the form:

$$\Gamma, \varphi \vdash_{\mathcal{L}} \psi \Longleftrightarrow \Gamma \vdash_{\mathcal{L}} \varphi' \bowtie \psi,$$

and the signatures of φ and φ' coincide.

Definition 1. *The relation $\vdash_{\mathcal{L}}$ satisfies the Craig interpolation property, if for each pair of formulas φ and ψ in signatures Σ_φ and Σ_ψ with the property $\varphi \vdash_{\mathcal{L}} \psi$, there exists a formula θ in signature $\Sigma_\varphi \cap \Sigma_\psi$ such that $\varphi \vdash_{\mathcal{L}} \theta$ and $\theta \vdash_{\mathcal{L}} \psi$.*

The relation $\vdash_{\mathcal{L}}$ is said to satisfy the strongest consequence property, if for every formula φ in signature Σ_φ and every subset $\Sigma \subseteq \Sigma_\varphi$ there exists a formula θ in signature Σ such that:

- *$\varphi \vdash_{\mathcal{L}} \theta$;*
- *if ψ is a formula in signature $\Sigma_\psi \subseteq \Sigma$ and $\varphi \vdash_{\mathcal{L}} \psi$, then $\theta \vdash \psi$.*

The formula θ from the conditions above is called the strongest consequence of φ in signature Σ.

Definition 2. *Let $\mathcal{T}$ be a set of formulas in signature Σ and $\Delta \subseteq \Sigma$ be its subsignature. We call $\mathcal{T}$ Δ–decomposable, if there exist non-empty sets $\mathcal{S}_1$ and $\mathcal{S}_2$ of formulas in signatures Σ_1 and Σ_2 such that $\Sigma_1 \cup \Sigma_2 = \Sigma$, $\Sigma_1 \cap \Sigma_2 = \Delta$, and $\mathcal{T} \sim_{\mathcal{L}} \mathcal{S}_1 \cup \mathcal{S}_2$.*

The pair $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle$ is called a Δ–decomposition of $\mathcal{T}$ and the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are called Δ–decomposition components of $\mathcal{T}$. We call a Δ–decomposition trivial, if $\Sigma_i = \Delta$ for some $i = 1, 2$.

If Δ is the empty set, then we have a decomposition into components with disjoint signatures. As the relation $\vdash_{\mathcal{L}}$ satisfies the tautology property, only non-trivial Δ–decompositions with $\Sigma_1 \neq \Delta \neq \Sigma_2$ are of interest for consideration. For instance, if the signature Σ consists of a single symbol, then every set of formulas in this signature has only trivial Δ–decompositions.

Definition 3. *We say that the relation $\vdash_{\mathcal{L}}$ satisfies the property of uniqueness of signature decompositions, if the following holds:*

for every set of formulas $\mathcal{T}$ in signature Σ and every subset $\Delta \subseteq \Sigma$ there exists a unique partition Π of the signature $\Sigma \setminus \Delta$ such that $\mathcal{T} \sim_{\mathcal{L}} \bigcup\{\mathcal{T}_\pi \mid \pi \in \Pi\}$, with every $\mathcal{T}_\pi$ a set of formulas in signature $\pi \cup \Delta$, which is equivalent to the set of all formulas in signature $\pi \cup \Delta$ entailed by $\mathcal{T}$, and has only trivial Δ–decompositions.

We further assume that in the case the partition Π is empty, $\bigcup\{\mathcal{T}_\pi \mid \pi \in \Pi\}$ is equivalent to the set $\mathcal{T}$.

3. Basic results

Theorem 1. *Assume that $\vdash_{\mathcal{L}}$ satisfies the Craig interpolation property. Then $\vdash_{\mathcal{L}}$ satisfies the property of uniqueness of signature decompositions.*

Proof. We suppose that the interpolation property is satisfied and prove three auxiliary lemmas.

Lemma 1. *Let Ψ_1 and Ψ_2 be sets of formulas in signatures Σ_1 and Σ_2, $\Sigma_1 \cup \Sigma_2 = \Sigma$, $\Sigma_1 \cap \Sigma_2 = \Delta$, and φ be a formula in signature Σ_φ.*

If $\Psi_1, \Psi_2 \vdash_{\mathcal{L}} \varphi$, then there exist formulas θ_1 and θ_2 in signatures Σ_1 and Σ_2 such that $\Psi_i \vdash_{\mathcal{L}} \theta_i$ (for $i = 1, 2$), $\theta_1, \theta_2 \vdash_{\mathcal{L}} \varphi$ and each θ_i for $i = 1, 2$ contains only those symbols of $\Sigma_i \setminus \Delta$ that occur in φ.

Proof of the lemma. As $\Psi_1, \Psi_2 \vdash_{\mathcal{L}} \varphi$, by the extensionality, compactness, and adjunction properties of $\vdash_{\mathcal{L}}$, there exist formulas ψ_1 and ψ_2 in signatures Σ_1 and Σ_2 such that $\Psi_i \vdash_{\mathcal{L}} \psi_i$ for $i = 1, 2$ and $\psi_1, \psi_2 \vdash_{\mathcal{L}} \varphi$. Then, by the deduction property of $\mathcal{L}$, we have $\psi_1 \vdash_{\mathcal{L}} \psi_2' \bowtie \varphi$, and the signatures of ψ_2' and ψ_2 coincide.

Note that ψ_1 and ψ_2' are formulas in signatures Σ_1, Σ_2 with $\Sigma_1 \cap \Sigma_2 \subseteq \Delta$ and φ is a formula in some signature Σ_φ. Thus, by the interpolation property, there exists a formula θ_1 in signature Σ_1 such that θ_1 contains only those symbols of $\Sigma_1 \setminus \Delta$ that occur in φ and satisfies $\psi_1 \vdash_{\mathcal{L}} \theta_1$ and $\theta_1 \vdash_{\mathcal{L}} \psi_2' \bowtie \varphi$. Hence, $\psi_2 \vdash_{\mathcal{L}} \theta_1' \bowtie \varphi$ and θ_1' is a formula in signature Σ_1, ψ_2 is a formula in Σ_2.

Similarly, there exists a formula θ_2 in signature Σ_2 such that θ_2 contains only those symbols of $\Sigma_2 \setminus \Delta$ that occur in φ and satisfies $\psi_2 \vdash_{\mathcal{L}} \theta_2$ and $\theta_2 \vdash_{\mathcal{L}} \theta_1' \bowtie \varphi$. By applying the deduction property of $\mathcal{L}$ again, we obtain formulas θ_1 and θ_2 for which $\theta_1, \theta_2 \vdash_{\mathcal{L}} \varphi$. $\square$

Based on Lemma 1, we now can introduce the following definition:

Definition 4. *Consider a set $\mathcal{T}$ of formulas in signature Σ and a subset $\Delta \subseteq \Sigma$. Let φ be a formula and Σ_φ be its signature. We call φ Δ–decomposable in $\mathcal{T}$, if there exist formulas θ_1 and θ_2 in signatures Σ_1 and Σ_2 such that $\Sigma_1 \cup \Sigma_2 \subseteq \Sigma$, $\Sigma_1 \cap \Sigma_2 \subseteq \Delta$, $\Sigma_1 \cap \Delta \neq \Sigma_1$, $\Sigma_2 \cap \Delta \neq \Sigma_2$, $(\Sigma_1 \cup \Sigma_2) \setminus \Delta \subseteq \Sigma_\varphi$, $\mathcal{T} \vdash_{\mathcal{L}} \theta_i$ (for i=1,2), and $\theta_1, \theta_2 \vdash_{\mathcal{L}} \varphi$.*

We call θ_1 and θ_2 Δ–decomposition fragments for φ in $\mathcal{T}$. If there are no such formulas θ_1 and θ_2 for φ, then we call φ Δ–indecomposable in $\mathcal{T}$.

Remark 1. *Let $\mathcal{T}$ be a set of formulas in signature Σ and $\Delta \subseteq \Sigma$ be a subsignature. Then, by Definition 4, each formula φ in signature $\Sigma_\varphi \subseteq \Delta$ is Δ–indecomposable in $\mathcal{T}$. In fact, from the condition $(\Sigma_1 \cup \Sigma_2) \setminus \Delta \subseteq \Sigma_\varphi$ we have in this case that $(\Sigma_1 \cup \Sigma_2) \setminus \Delta = \varnothing$; thus, $\Sigma_1 \cup \Sigma_2 \subseteq \Delta$ and $\Sigma_i \cap \Delta = \Sigma_i$ for $i = 1, 2$, while the latter contradicts the conditions from Definition 4.*

Lemma 2. *Consider a set $\mathcal{T}$ of formulas in signature Σ and a formula φ such that $\mathcal{T} \vdash_{\mathcal{L}} \varphi$. For every subsignature $\Delta \subseteq \Sigma$ there exists a sequence of formulas $\theta_1, \ldots, \theta_n$, with each θ_i, $i = 1, .., n$ Δ–indecomposable in $\mathcal{T}$, such that $\mathcal{T} \vdash_{\mathcal{L}} \theta_i$ for $i = 1, .., n$ and $\theta_1, \ldots, \theta_n \vdash_{\mathcal{L}} \varphi$.*

Proof of the lemma. Consider the set $T_1 = \{\varphi\}$. Take the Δ–decomposition fragments ξ and ψ for φ, if they exist in $\mathcal{T}$, and build the set $T_2 = \{\xi, \psi\}$. By repeating this transformation for the formulas of T_2 and further resulting sets, we obtain the sequence $T_1, T_2, T_3, \ldots$. Each formula of $\mathcal{L}$ is of finite length, therefore, contains only finitely many signature symbols and can be decomposed only finitely many times. Thus, for some k, the set $T_k = \{\theta_1, \ldots, \theta_n\}$ will contain only those formulas that are Δ–indecomposable in $\mathcal{T}$, and for which, by the transitivity property of $\vdash_{\mathcal{L}}$, we have $\theta_1, \ldots, \theta_n \vdash_{\mathcal{L}} \varphi$. $\square$

Lemma 3. *Let $\mathcal{T}$ be a set of formulas in signature Σ, $\Delta \subseteq \Sigma$, and $\mathcal{S}_1$ be equivalent to the set of all formulas in signature $\Sigma_1 \subseteq \Sigma$ entailed by $\mathcal{T}$, $\Delta \subseteq \Sigma_1$. Then $\mathcal{S}_1$ is a Δ–decomposition component of $\mathcal{T}$ if and only if the following condition is satisfied:*

(*) for every formula φ, if $\mathcal{T} \vdash_{\mathcal{L}} \varphi$, $\Sigma_\varphi \subseteq \Sigma$ is the signature of φ, $\Sigma_\varphi \cap (\Sigma_1 \setminus \Delta) \neq \varnothing$, and φ is Δ–indecomposable in $\mathcal{T}$, then $\Sigma_\varphi \subseteq \Sigma_1$.

Proof of the lemma.

$\Rightarrow$: Let $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle$ be a Δ–decomposition of $\mathcal{T}$, where $\mathcal{S}_2$ is a set of formulas in signature $\Sigma_2 = \Delta \cup (\Sigma \setminus \Sigma_1)$. Let φ be a formula such that $\mathcal{T} \vdash_{\mathcal{L}} \varphi$, $\Sigma_\varphi \subseteq \Sigma$ is the signature of φ, $\Sigma_\varphi \cap (\Sigma_1 \setminus \Delta) \neq \varnothing$, φ is Δ–indecomposable in $\mathcal{T}$, but $\Sigma_\varphi \not\subseteq \Sigma_1$. Then $\Sigma_\varphi \cap (\Sigma_2 \setminus \Delta) \neq \varnothing$ and, by Lemma 1, from $\mathcal{S}_1, \mathcal{S}_2 \vdash_{\mathcal{L}} \varphi$ we obtain that φ is Δ–decomposable in $\mathcal{T}$; contradiction.

$\Leftarrow$: Let $\mathcal{S}_2$ be a set of all formulas in signature $\Sigma_2 = \Delta \cup (\Sigma \setminus \Sigma_1)$ entailed by $\mathcal{T}$. By the tautology property of $\vdash_{\mathcal{L}}$, this set is non-empty. We may also assume that each symbol of Δ occurs in the formulas of $\mathcal{S}_2$ (as well as in the formulas of $\mathcal{S}_1$). Let ψ be a formula in signature Σ entailed by $\mathcal{T}$. Then, by Lemma 2, there exists a sequence of formulas $\theta_1, \ldots, \theta_n$, with each θ_i, $i = 1, .., n$ Δ–indecomposable in $\mathcal{T}$, such that $\theta_1, \ldots, \theta_n \vdash_{\mathcal{L}} \psi$. According to Definition 4, the premise of the condition $(*)$ is then satisfied for every θ_i, $i = 1, .., n$ and an appropriate set of formulas $\mathcal{S}_j$, $j = 1, 2$. Thus, the signature of each θ_i is contained either in Σ_1, or in Σ_2. Hence, by the definition of $\mathcal{S}_j$, $j = 1, 2$, we have $\mathcal{S}_1 \cup \mathcal{S}_2 \vdash_{\mathcal{L}} \{\theta_1, \ldots, \theta_n\}$ and, by transitivity of $\vdash_{\mathcal{L}}$, we obtain $\mathcal{S}_1 \cup \mathcal{S}_2 \vdash_{\mathcal{L}} \psi$. As ψ is an arbitrary formula entailed by $\mathcal{T}$, we conclude that $\mathcal{T} \sim_{\mathcal{L}} \mathcal{S}_1 \cup \mathcal{S}_2$ and $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle$ is a Δ–decomposition of $\mathcal{T}$. $\square$

We are now ready to complete the proof of Theorem 1. Let $\mathcal{T}$ be a set of formulas in signature Σ and $\Delta \subseteq \Sigma$ be a subsignature. Consider the set of all subsets of Σ that contain Δ; denote this set by Ω. By the tautology property of $\vdash_{\mathcal{L}}$, for each $\Sigma_1 \in \Omega$ there is a corresponding non-empty set of formulas in signature Σ_1 entailed by $\mathcal{T}$. It follows from Lemma 3 that a subset $\Sigma_1 \in \Omega$ corresponds to a Δ–decomposition component of $\mathcal{T}$, which has only trivial Δ–decompositions, iff Σ_1 satisfies $(*)$ and does not have a proper subset satisfying $(*)$. Note that the collection of sets from Ω with the property $(*)$ is closed under intersection; thus, each symbol of Σ is contained in one minimal set from Ω satisfying (*), and the intersection of these sets is Δ. Thus, the property of uniqueness of signature decompositions is proved. $\square$

The following fact follows immediately from the proof of Theorem 1:

Corollary 1. *Let $\vdash_{\mathcal{L}}$ satisfy the Craig interpolation property, $\mathcal{T}$ be a set of formulas in signature Σ, and $\Delta \subseteq \Sigma$. Consider the set Ω of all Δ–decomposition components of $\mathcal{T}$ with the relation $\prec \subseteq \Omega \times \Omega$ defined as follows: for every $\mathcal{S} \in \Omega$ and $\mathcal{U} \in \Omega$, we have $\mathcal{S} \prec \mathcal{U}$ iff $\mathcal{S}$ is a Δ–decomposition component of $\mathcal{U}$.*

Then $(\Omega, \prec)$ is a boolean algebra with the infimum equal to the set of all formulas in signature Δ entailed by $\mathcal{T}$ and the supremum equal to the set of all formulas of $\mathcal{T}$.

Let us introduce the following auxiliary definition:

Definition 5. *Let $\mathcal{T}$ be a set of formulas in signature Σ and $\Delta \subseteq \Sigma$. We call the union of sets $\bigcup\{\mathcal{T}_\pi \mid \pi \in \Pi\}$, equivalent to $\mathcal{T}$, the canonical Δ–decomposition for $\mathcal{T}$, if Π is a partition of the signature $\Sigma \setminus \Delta$ and every $\mathcal{T}_\pi$ is a set of formulas in signature $\pi \cup \Delta$, which is equivalent to the set of all formulas in signature $\pi \cup \Delta$ entailed by $\mathcal{T}$ and has only trivial Δ–decompositions.*

We now describe for $\vdash_{\mathcal{L}}$ more precisely the connection between the interpolation property and the property of uniqueness of signature decompositions.

Proposition 1. *The relation $\vdash_{\mathcal{L}}$ satisfies the Craig interpolation property if and only if $\vdash_{\mathcal{L}}$ satisfies the property of uniqueness of signature decompositions and the following additional property:*

(**) Let $\mathcal{T}$ and $\mathcal{T}'$ be two equivalent sets of formulas in signatures Σ and $\Sigma' \supseteq \Sigma$, respectively, $\Delta \subseteq \Sigma$, and the set $\bigcup\{\mathcal{T}_\pi \mid \pi \in \Pi\}$ be the canonical Δ–decomposition for $\mathcal{T}$. Then the set $\cup\{\mathcal{T}_\pi \mid \pi \in \Pi\} \bigcup \cup\{\mathcal{T}_\varepsilon \mid \varepsilon \in \Upsilon\}$ is the canonical Δ–decomposition for $\mathcal{T}'$, where Υ can be chosen as the partition of the signature $\Sigma' \setminus \Sigma$ into one-element subsets and every $\mathcal{T}_\varepsilon$ can be chosen equivalent to the set of all formulas in signature Δ entailed by $\mathcal{T}$.

Proof.

$\Rightarrow$: According to Theorem 1, it suffices to prove that the interpolation property yields $(**)$. Let $\mathcal{T}$ and $\mathcal{T}'$ be equivalent sets of formulas in signatures Σ and $\Sigma' \supseteq \Sigma$, respectively, and $\Delta \subseteq \Sigma$. By Theorem 1, there exists the canonical Δ–decomposition $\bigcup\{\mathcal{T}_\pi \mid \pi \in \Pi\}$ for $\mathcal{T}$, where Π is a partition of the signature $\Sigma \setminus \Delta$.

Let $\mathcal{P}$ be the set of all formulas in signature $(\Sigma' \setminus \Sigma) \cup \Delta$ entailed by $\mathcal{T}'$. Consider the union $\bigcup\{\mathcal{P}_\varepsilon \mid \varepsilon \in \Upsilon\}$, where Υ is the partition of the signature $\Sigma' \setminus \Sigma$ into one-element subsets and every $\mathcal{P}_\varepsilon$ is a set of formulas in signature $\varepsilon \cup \Delta$ equivalent to the set of all formulas in $\varepsilon \cup \Delta$ entailed by $\mathcal{T}$. We have $\mathcal{P} \vdash_{\mathcal{L}} \bigcup\{\mathcal{P}_\varepsilon \mid \varepsilon \in \Upsilon\}$. On the other hand, for each formula φ in signature $(\Sigma' \setminus \Sigma) \cup \Delta$, if $\mathcal{T}' \vdash_{\mathcal{L}} \varphi$, then, by the interpolation property, there exists a formula θ in signature Δ such that $\mathcal{T} \vdash_{\mathcal{L}} \theta$ and $\theta \vdash_{\mathcal{L}} \varphi$. Therefore, $\mathcal{P} \sim_{\mathcal{L}} \bigcup\{\mathcal{P}_\varepsilon \mid \varepsilon \in \Upsilon\}$ and each $\mathcal{P}_\varepsilon$ is equivalent to the set of all formulas in signature Δ entailed by $\mathcal{T}$.

We obtain that $\cup\{\mathcal{T}_\pi \mid \pi \in \Pi\} \bigcup \cup\{\mathcal{P}_\varepsilon \mid \varepsilon \in \Upsilon\}$ is the canonical Δ–decomposition for $\mathcal{T}'$ satisfying the conditions from $(**)$.

$\Leftarrow$: Let φ and ψ be formulas in signatures Σ_φ and Σ_ψ such that $\varphi \vdash_{\mathcal{L}} \psi$. Denote $\mathcal{T} = \{\varphi\}$, $\mathcal{T}' = \{\varphi, \psi\}$, $\Delta = \Sigma_\varphi \cap \Sigma_\psi$; then $\mathcal{T} \sim_{\mathcal{L}} \mathcal{T}'$.

Let $\cup\{\mathcal{T}_\pi \mid \pi \in \Pi\}$ be the canonical Δ–decomposition for $\mathcal{T}$. Then, by the $(**)$ property, the union $\cup\{\mathcal{T}_\pi \mid \pi \in \Pi\} \bigcup \cup\{\mathcal{T}_\varepsilon \mid \varepsilon \in \Upsilon\}$ is the canonical Δ–decomposition for $\mathcal{T}'$, where Υ is the partition of $\Sigma_\psi \setminus \Delta$ into one-element subsets and every $\mathcal{T}_\varepsilon$ is equivalent to the set of all formulas in signature Δ entailed by $\mathcal{T}$.

Consider the set $\mathcal{P}$ of all formulas in signature Σ_ψ entailed by φ. Let $\bigcup\{\mathcal{P}_\lambda \mid \lambda \in \Lambda\}$ be the canonical Δ–decomposition for $\mathcal{P}$, where Λ is a partition of the signature $\Sigma_\psi \setminus \Delta$. Then $\cup\{\mathcal{T}_\pi \mid \pi \in \Pi\} \bigcup \cup\{\mathcal{P}_\lambda \mid \lambda \in \Lambda\}$ is the canonical Δ–decomposition for $\mathcal{T}'$. Besides, $\cup\{\mathcal{P}_\lambda \mid \lambda \in \Lambda\} \vdash_{\mathcal{L}} \mathcal{P}$ and, by the transitivity property of $\vdash_{\mathcal{L}}$, we have $\cup\{\mathcal{P}_\lambda \mid \lambda \in \Lambda\} \vdash_{\mathcal{L}} \psi$.

Thus, we have two canonical Δ–decompositions for $\mathcal{T}'$. By the property of uniqueness of signature decompositions, we conclude that the partitions Υ and Λ of the signature $\Sigma_\psi \setminus \Delta$ coincide, hence, by the definition of the canonical Δ–decomposition, we have $\cup\{\mathcal{T}_\varepsilon \mid \varepsilon \in \Upsilon\} \sim_{\mathcal{L}} \cup\{\mathcal{P}_\lambda \mid \lambda \in \Lambda\}$; thus, $\cup\{\mathcal{T}_\varepsilon \mid \varepsilon \in \Upsilon\} \vdash_{\mathcal{L}} \psi$. By $(**)$, each $\mathcal{T}_\varepsilon$ can be chosen equivalent to the set of all formulas in signature Δ entailed by $\mathcal{T}$. Hence, by the transitivity, compactness, extensionality, and adjunction properties of $\vdash_{\mathcal{L}}$, we obtain a formula θ in signature Δ such that $\varphi \vdash_{\mathcal{L}} \theta$ and $\theta \vdash_{\mathcal{L}} \psi$. $\square$

Definition 6. *The Δ–decomposability property is said to be decidable in a calculus $\mathcal{L}$, if there exists an effective procedure to decide for every finite set $\mathcal{T}$ of formulas in a finite signature Σ and any given subsignature $\Delta \subseteq \Sigma$, whether $\mathcal{T}$ has a non-trivial Δ–decomposition.*

Note that in the case the relation $\vdash_{\mathcal{L}}$ is decidable and satisfies the extensionality and compactness properties and for a given finite set $\mathcal{T}$ of formulas in finite signature Σ and a subsignature $\Delta \subseteq \Sigma$ it is known that $\mathcal{T}$ has a non-trivial Δ–decomposition, then one can effectively build finite sets $\mathcal{S}_1$ and $\mathcal{S}_2$ of formulas such that $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle$ is a non-trivial Δ–decomposition of $\mathcal{T}$.

Proposition 2. *The Δ–decomposability property is decidable in a calculus $\mathcal{L}$, if the corresponding relation $\vdash_{\mathcal{L}}$ satisfies the extensionality, transitivity, and adjunction properties and meets the following additional conditions:*

- *$\vdash_{\mathcal{L}}$ is decidable;*
- *$\vdash_{\mathcal{L}}$ satisfies the strongest consequence property;*
- *there exists an effective procedure to find the strongest consequence for every formula $\varphi \in \mathcal{L}$ in signature Σ and every subset $\Sigma' \subseteq \Sigma$.*

Proof. Let $\mathcal{T}$ be a finite set of formulas in a finite signature Σ. Then, by the extensionality and adjunction properties, the set $\mathcal{T}$ is equivalent in $\mathcal{L}$ to some formula φ in signature Σ.

Suppose that for some $\Delta \subseteq \Sigma$, the set $\mathcal{T}$ is Δ–decomposable with the components $\mathcal{S}_1$ and $\mathcal{S}_2$ in signatures Σ_1 and Σ_2. Let θ_1 and θ_2 be the strongest consequences of φ in signatures Σ_1 and Σ_2, respectively. Then $\{\theta_i\} \vdash_{\mathcal{L}} \mathcal{S}_i$ for $i = 1, 2$ and, by the transitivity property of $\vdash_{\mathcal{L}}$, we have $\{\theta_1, \theta_2\} \vdash \mathcal{T}$. As $\mathcal{T} \vdash_{\mathcal{L}} \theta_i$, $i = 1, 2$, we obtain that $\mathcal{T}$ is Δ–decomposable with the components $\{\theta_1\}$ and $\{\theta_2\}$.

Thus, $\mathcal{T}$ has a non-trivial Δ–decomposition iff $\{\varphi\} \sim_{\mathcal{L}} \{\theta_1, \theta_2\}$, where θ_1 and θ_2 are the strongest consequences of φ in some signatures Σ_1 and Σ_2 satisfying $\Sigma_1 \cup \Sigma_2 = \Sigma$, $\Sigma_1 \cap \Sigma_2 = \Delta$, and $\Sigma_1 \neq \Delta \neq \Sigma_2$. As the relation $\vdash_{\mathcal{L}}$ is computable and there exists an effective procedure to find strongest consequences, we obtain the decidability of the Δ–decomposability property in $\mathcal{L}$. $\square$

4. Summary

Note that the restrictions formulated for $\vdash_{\mathcal{L}}$ in Section 2 are rather standard and satisfied in many calculi, including a wide class of modal logics. The Craig interpolation property is one of the most important properties when studying logics and there is a significant number of calculi known to have it. Due to Theorem 1 and the restrictions on $\vdash_{\mathcal{L}}$, we can state that the property of uniqueness of signature decompositions is usually satisfied in those calculi for which the interpolation property is proved. Without going into details, we refer the reader to the paper [12] and monograph [5] containing a summary of results on interpolation.

¿From the decidability point of view, even more important for the Δ–decomposability property is uniform interpolation property, which is also well-studied in modal [14, 6, 4] and description [8] logics. It yields Craig interpolation and generalizes the strongest consequence property formulated in Section 2. The use of uniform interpolation in building algorithms is somewhat restricted due to unacceptable computational bounds proved for this property in various calculi. Nevertheless, in many cases, studying uniform interpolation allows to quickly answer a number of questions related to the Δ–decomposability property. Due to Theorem 1, Proposition 2 and the restrictions on $\vdash_{\mathcal{L}}$ defined in Section 2, the majority of calculi known to have uniform interpolation also satisfies the property of uniqueness of signature decompositions and has the decidable Δ–decomposability property. The examples of these calculi are intuitionistic logic, modal logics Grz, GL, S5, K and the modal μ-calculus.

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