Bull. Nov. Comp. Center, Math. Model. in Geoph., 17 (2014), 35–45

On divergence representations of the Gaussian and the mean curvature of surfaces and applications*

A.G. Megrabov

Abstract. The new meaning or a property of the Gaussian and of the mean curvature of surfaces forming a family in terms of the vector analysis has been discovered. The divergence representations were found for the mean curvature $H = H(x,y,z)$ and the Gaussian curvature of $K = K(x,y,z)$ of the surfaces S_α, given either by the equation $u(x,y,z) = \alpha$ (α is a parameter, u is a scalar function) or parametrically; or the surfaces S_τ that are described by some general properties. The surfaces S_α and S_τ continuously fill the domain D, forming a family of $\{S_\alpha\}$ or $\{S_\tau\}$ in D with the unit normal field $\boldsymbol{\tau} = \boldsymbol{\tau}(x,y,z)$. Thus, the formulas of the form $H = \operatorname{div} \boldsymbol{S}_H$, $K = \operatorname{div} \boldsymbol{S}_K$ are obtained, and three-dimensional vector fields $\boldsymbol{S}_H$, $\boldsymbol{S}_K$ are expressed in terms of the normal field $\boldsymbol{\tau}$: $\boldsymbol{S}_H = -\frac{1}{2}\boldsymbol{\tau}$, $\boldsymbol{S}_K = -\frac{1}{2}\boldsymbol{S}(\boldsymbol{\tau})$, $\boldsymbol{S}(\boldsymbol{\tau}) = \operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{\tau} - \boldsymbol{\tau} \operatorname{div} \boldsymbol{\tau}$ and have a clear geometric meaning. In the case when the surface is given by the graph $z = f(x,y)$, the above formulas lead to divergent representations for the mean and Gaussian curvatures obtained earlier.

The applications of these general geometric formulas to the equations of mathematical physics: the eikonal equation, the Poisson equation, Euler's hydrodynamic equations are given. Furthermore, in the plane case a simple geometric interpretation of the conservation laws obtained earlier for a family of plane curves and for solutions to the eikonal and Euler's hydrodynamic equations is given.

This paper is a continuation and development of works [1–5].

Important characteristics of a surface in the classical differential geometry [6–12] are: its unit normal $\boldsymbol{\tau}$, the principal directions $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$, the principal curvatures k_1 and k_2, the mean curvature $H \overset{\text{def}}{=} (k_1 + k_2)/2$ and the Gaussian curvature $K \overset{\text{def}}{=} k_1 k_2$, defined at each point (x,y,z) of a given surface. The vector physical fields, described by the equations of mathematical physics, have the vector lines L_τ (e.g., rays or stream lines) forming the family of curves $\{L_\tau\}$ and continuously filling the domain D. The surfaces S_τ with the unit normal $\boldsymbol{\tau}$, which are orthogonal to these curves L_τ (e.g., wave fronts), also form the family $\{S_\tau\}$. Therefore, in this paper we consider not only the properties of the fixed surface S_τ, but also the properties of the family surfaces $\{S_\tau\}$, continuously filling some domain D in the space with the Cartesian coordinates x, y, z. And the characteristics of $\boldsymbol{\tau}$, $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, k_1, k_2, H, K of the surfaces S_τ are a three-dimensional vector and scalar fields (the functions of x, y, z) in D.

*Supported by RFBR under Grant 12-01-00648.

For the Gaussian curvature K of the surface two classical definitions or meanings are known: 1) K is the product of the principal curvatures — $K \stackrel{\text{def}}{=} k_1 k_2$ [6–12]; 2) K is the limit of the ratio of the area of the spherical mapping of the surface domain onto the area of the domain, when the domain is contracted to the point [10]. For the mean curvature H of the surface, we have the classical definition as half-sum of the principal curvatures — $H \stackrel{\text{def}}{=} \frac{1}{2}(k_1 + k_2)$ [6–12].

In this paper, we establish another meaning or a property of the Gaussian and the mean curvature in terms of the vector analysis. It turns out that if the surfaces form a family and continuously fill a certain domain in the three-dimensional space, then with general conditions, the Gaussian $K = K(x, y, z)$, and the mean curvature $H = H(x, y, z)$ of every surface of this family at each point (x, y, z) is the divergence (the sources density in terms of the field theory) of a vector field with a certain geometric sense. More precisely, we obtain the divergence representations for the mean curvature H and the Gaussian curvature K of the surfaces S_α, given by the equation $u(x, y, z) = \alpha$ (α is a parameter) or, parametrically, and of the surfaces S_τ, which are described by some general properties. The surfaces S_α and S_τ form the family $\{S_\alpha\}$ and $\{S_\tau\}$ in the domain D and have the field of unit normals $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z)$. Thus, the formulas of the form $H = \operatorname{div} \boldsymbol{S}_H$, $K = \operatorname{div} \boldsymbol{S}_K$ are obtained, and the three-dimensional vector fields $\boldsymbol{S}_H$, $\boldsymbol{S}_K$ are expressed in terms of the field of unit normals $\boldsymbol{\tau}$.

It is important that the vector fields $\boldsymbol{S}_H$ and $\boldsymbol{S}_K$ have a clear geometric meaning, that is, $\boldsymbol{S}_H$ and $\boldsymbol{S}_K$ are expressed in terms of the normal $\boldsymbol{\tau}$ by the formulas $\boldsymbol{S}_H = -\frac{1}{2}\boldsymbol{\tau}$, $\boldsymbol{S}_K = -\frac{1}{2}\boldsymbol{S}(\boldsymbol{\tau})$, $\boldsymbol{S}(\boldsymbol{\tau}) = \operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{\tau} - \boldsymbol{\tau} \operatorname{div} \boldsymbol{\tau}$. The geometric meaning of the vector $\boldsymbol{S}_H$ is obvious, and the vector $\boldsymbol{S}(\boldsymbol{\tau})$ is the sum of the three vectors of curvature of the three mutually orthogonal curves (at each point $(x, y, z) \in D$): of the curvature vector of the vector line L_τ of the normal field $\boldsymbol{\tau}$ and of two curvature vectors of the two geodesic lines on the surface S_α or S_τ, with any two mutually orthogonal directions at this point. From these representations the relationships between the properties of the surfaces and the properties of the fields $\boldsymbol{\tau}$, $\boldsymbol{S}(\boldsymbol{\tau})$ and some integral formulas follow.

In the case when the surface is defined by the graph $z = f(x, y)$, the obtained formulas reduce to the divergence representations for the mean curvature $H(x, y)$ [7, p. 92] and the Gaussian curvature $K(x, y)$ [1].

The applications of these general geometric formulas to the equations of the mathematical physics — the eikonal equation, Euler's hydrodynamic equations, and Poisson's equation — are obtained. In this consideration, the role of the curves L_τ is played by the vector lines of the corresponding vector fields (solutions of the equation), for example, the rays for solutions of the eikonal equation, and the role of the surfaces S_α and S_τ is played by the surfaces which are orthogonal to these curves (e. g., the wave fronts for the eikonal equation). The formulas for the mean and the Gaussian curvatures of the surfaces generated by the solutions of these equations in terms of these solutions are obtained. Furthermore, in the plane case, a simple geometric interpretation of the conservation laws, obtained in [2–5] for a family of plane curves and for the solutions to the eikonal equation and Euler's hydrodynamic equations is given.

The symbols $(\boldsymbol{a}\cdot\boldsymbol{b})$ and $\boldsymbol{a}\times\boldsymbol{b}$ denote the scalar and the vector product of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, ∇ is the Hamiltonian operator, $(\boldsymbol{v}\cdot\nabla)\boldsymbol{a}$ is the derivative of the vector $\boldsymbol{a}$ in the direction of the unit vector $\boldsymbol{v}$, $\Delta u = u_{xx} + u_{yy} + u_{zz}$; $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are the unit vectors along the axes x, y, z of the Cartesian coordinate system.

1. In [2], there is obtained

Lemma 1. *For any vector field* $\boldsymbol{v} = \boldsymbol{v}(x, y, z) = v_1\boldsymbol{i}+v_2\boldsymbol{j}+v_3\boldsymbol{k} = |\boldsymbol{v}|\boldsymbol{\tau}$ *with the definition domain* D, *the components* $v_k(x, y, z) \in C^1(D)$, $k = 1, 2, 3$, *the modulus* $|\boldsymbol{v}| \neq 0$ *in* D *and the direction* $\boldsymbol{\tau} = \boldsymbol{v}/|\boldsymbol{v}|$ $(|\boldsymbol{\tau}| \equiv 1)$ *in* D *the following identity is valid*:

$$\boldsymbol{T}(\boldsymbol{v}) = \boldsymbol{S}(\boldsymbol{\tau}), \tag{1}$$

where

$$\boldsymbol{S}(\boldsymbol{\tau}) \stackrel{\text{def}}{=} \operatorname{rot}\boldsymbol{\tau}\times\boldsymbol{\tau} - \boldsymbol{\tau}\operatorname{div}\boldsymbol{\tau} = (\boldsymbol{\tau}\cdot\nabla)\boldsymbol{\tau} - \boldsymbol{\tau}\operatorname{div}\boldsymbol{\tau}, \tag{2}$$

$$\boldsymbol{T}(\boldsymbol{v}) \stackrel{\text{def}}{=} \operatorname{grad}\ln|\boldsymbol{v}| + \{\operatorname{rot}\boldsymbol{v}\times\boldsymbol{v} - \boldsymbol{v}\operatorname{div}\boldsymbol{v}\}/|\boldsymbol{v}|^2. \tag{3}$$

Also, $\boldsymbol{S}(\boldsymbol{\tau}) = \boldsymbol{K}_\tau - \boldsymbol{\tau}\operatorname{div}\boldsymbol{\tau} = k\boldsymbol{\nu} - \boldsymbol{\tau}\operatorname{div}\boldsymbol{\tau}$, *where* $\boldsymbol{K}_\tau \stackrel{\text{def}}{=} k\boldsymbol{\nu} = (\boldsymbol{\tau}\cdot\nabla)\boldsymbol{\tau} = \operatorname{rot}\boldsymbol{\tau}\times\boldsymbol{\tau}$ *is the curvature vector of the vector line* L_τ *of the field* $\boldsymbol{\tau}$ *or* $\boldsymbol{v}$, k *and* $\boldsymbol{\nu}$ *are its curvature and the unit principal normal.*

First, we consider the case when the surface is given by the equation

$$u(x, y, z) = \alpha. \tag{4}$$

The surface defined by the graph $z = f(x, y)$ corresponds to the case $u(x, y, z) = z - f(x, y)$. A set of the level surfaces S_α of the function $u(x, y, z)$ of the form (4) in its definition domain D forms the family $\{S_\alpha\}$ with a real parameter $\alpha \in [\alpha_1, \alpha_2]$. The tangent unit vector $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z) = \operatorname{grad} u/|\operatorname{grad} u|$ (direction) of the vector lines of the field $\operatorname{grad} u$ is at the same time the normal to the surface S_α, passing through the point (x, y, z) (under condition (4)).

The direct calculation of the expressions for $\operatorname{div}\boldsymbol{\tau}$ and $\operatorname{div}\boldsymbol{S}(\boldsymbol{\tau})$ leads to

Lemma 2. *Let the scalar function* $u(x,y,z)$ *be defined in* D, $u(x,y,z) \in C^k(D)$ (*below* $k=2$ *or* $k=3$) $|\operatorname{grad} u| \neq 0$ *in* D, $\boldsymbol{\tau} = \boldsymbol{\tau}(x,y,z) = \operatorname{grad} u/|\operatorname{grad} u|$ *is the field direction of the field* $\boldsymbol{v} = \operatorname{grad} u$. *Then we have in* D: $\operatorname{div}\boldsymbol{\tau} = U_H(u) \overset{\text{def}}{=} |\operatorname{grad} u|^{-3}\{(u_y^2+u_z^2)u_{xx} + (u_x^2+u_z^2)u_{yy} + (u_x^2+u_y^2)u_{zz} - 2(u_x u_y u_{xy} + u_x u_z u_{xz} + u_y u_z u_{yz})\}$ *for* $k=2$; $\operatorname{div}\boldsymbol{S}(\boldsymbol{\tau}) = \operatorname{div}\boldsymbol{T}(u) = U_K(u) \overset{\text{def}}{=} -2|\operatorname{grad} u|^{-4}\{u_z^2(u_{xx}u_{yy} - u_{xy}^2) + u_y^2(u_{xx}u_{zz} - u_{xz}^2) + u_x^2(u_{yy}u_{zz} - u_{yz}^2) + 2[U_y u_{xy}(u_z u_{xz} - u_x u_{zz}) + u_x u_{xz}(u_y u_{yz} - u_z u_{yy}) + u_z u_{yz}(u_x u_{xy} - u_y u_{xx})]\}$ *for* $k=3$; $\boldsymbol{S}(\boldsymbol{\tau}) = \boldsymbol{T}(u) \overset{\text{def}}{=} \operatorname{grad}\ln|\operatorname{grad} u| - \Delta u \operatorname{grad} u/|\operatorname{grad} u|^2$.

Lemma 3. *Let for the function* $u(x,y,z)$, *which determines the surface* S_α *by equation* (4), *the conditions of Lemma 2 be satisfied and* $u(x,y,z) \in C^2(D)$. *Let* H *and* K *be the mean and the Gaussian curvature at the point* (x,y,z) *for the surface* S_α *passing through this point. Then* H *and* K *are expressed in terms of the derivatives of the function* u *by the formulas* $H = -\dfrac{1}{2}U_H(u)$, $K = -\dfrac{1}{2}U_K(u)$, *where the expressions* $U_H(u)$, $U_K(u)$ *are defined in Lemma 2 and calculated at the same point* (x,y,z).

Proof. In the case of the surface $z = f(x,y)$ for the coefficients E, F, G and L, M, N, respectively, of its first and the second quadratic form we have the known formulas $E = 1 + f_x^2$, $F = f_x f_y$, $G = 1 + f_y^2$, $L = f_{xx}/\sqrt{1+g}$, $M = f_{xy}/\sqrt{1+g}$, $N = f_{yy}/\sqrt{1+g}$, $g = f_x^2 + f_y^2$ [6–12]. Considering equation (4) as the implicit definition of the function $z = f(x,y)$, i. e. as the identity $u(x,y,f(x,y)) = \alpha$, and differentiating it with respect to x and y, we express the derivatives f_x, f_y, f_{xx}, f_{xy}, f_{yy} in terms of the derivatives of the function $u(x,y,z)$, e. g. $f_x = -u_x/u_z$, $f_y = -u_y/u_z$, $f_{xx} = -(u_z^2 u_{xx} - 2u_x u_z u_{xz} + u_x^2 u_{zz})/u_z^3$, etc. These formulas are independent of α. Without loss of generality, we assume that $u_z \neq 0$ since $|\operatorname{grad} u| \neq 0$. Substituting these expressions into the formulas for E, F, G, L, M, N and applying the known formulas [6–12] $H = \frac{1}{2}(LG - 2MF + NE)/(EG - F^2)$, $K = (LN - M^2)/(EG - F^2)$ for H, K, we obtain the expressions of the lemma. □

In the case of the surface $z = f(x,y)$, the known formula [7, p. 90] $K = (f_{xx}f_{yy} - f_{xy}^2)/(1 + f_x^2 + f_y^2)^2$ follows from Lemma 3. The comparison of the expressions of Lemmas 2, 3 for $\operatorname{div}\boldsymbol{\tau}$ and H, $\operatorname{div}\boldsymbol{S}(\boldsymbol{\tau})$ and K implies

Theorem 1. *Let the scalar function* $u(x,y,z)$ *be defined in* D, $u(x,y,z) \in C^k(D)$, $|\operatorname{grad} u| \neq 0$ *in* D, $\boldsymbol{\tau} = \boldsymbol{\tau}(x,y,z) = \operatorname{grad} u/|\operatorname{grad} u|$ *is the vector field of the unit normals to the surfaces* $S_\alpha \in \{S_\alpha\}$ *of the form* (4). *Then, at any point* $(x,y,z) \in D$, *the mean curvature* H *for* $k=2$ *and the Gaussian curvature* K *for* $k=3$ *of the surface* S_α *of the form* (4) *passing through this point are respectively the divergence* (*the sources density*) *of the vector* *field* $\boldsymbol{S}_H \stackrel{\text{def}}{=} -\boldsymbol{\tau}/2$ *and the divergence of the vector field* $\boldsymbol{S}_K \stackrel{\text{def}}{=} -\boldsymbol{S}(\boldsymbol{\tau})/2 = -\boldsymbol{T}(u)/2$ *at this point, where the vector fields* $\boldsymbol{S}(\boldsymbol{\tau})$, $\boldsymbol{T}(u)$ *are defined in* (2) *and in Lemma 2. Thus, in* D, *the following identities hold*:

$$H = -\frac{1}{2}\,\mathrm{div}\,\boldsymbol{\tau}, \tag{5}$$

$$K = -\frac{1}{2}\,\mathrm{div}\,\boldsymbol{S}(\boldsymbol{\tau}) \quad \Leftrightarrow \tag{6}$$

$$K = -\frac{1}{2}\,\mathrm{div}\,\boldsymbol{T}(u). \tag{7}$$

If the values H, K *in the left-hand side in* (5)–(7) *are defined on the fixed surface* S_α, *then the quantities in the right-hand side are also calculated on this surface* S_α, *i.e., under condition* (4).

Corollary 1. *In the case of the surface* $z = f(x,y)$ *for its Gaussian curvature* $K(x,y)$ *from Theorem 1 follows the formula* $K = \mathrm{div}\,\boldsymbol{V}_1$, $\boldsymbol{V}_1 = -g\boldsymbol{S}(\boldsymbol{\tau}_0)/2(1+g) = -g\boldsymbol{T}_0(f)/2(1+g)$ *[1], where* $g = f_x^2 + f_y^2$, $\boldsymbol{\tau}_0 = \mathrm{grad}\, f/g^{1/2}$ *is the direction of the field* $\mathrm{grad}\, f$, $\boldsymbol{T}_0 = \boldsymbol{T}(f)$, *and for the mean curvature* $H(x,y)$ *follows the formula* $H = \mathrm{div}\{\mathrm{grad}\, f/(1+g)^{1/2}\}$ *[7, p. 92].*

Corollary 2. *Let the surface* S_α *be defined by the parametric equations* $\boldsymbol{r} = \boldsymbol{r}(u,v,\alpha)$, *where* $\boldsymbol{r} = (x,y,z)$ *is the radius vector,* u *and* v *are the parameters for the surface* S_α, $(u,v) \in D'$, D' *is a domain on the plane* u, v, *and the surface* S_α *for any* $\alpha \in I \stackrel{\text{def}}{=} [\alpha_1, \alpha_2]$ *is* C^k*-regular [12], so that* $\boldsymbol{r}(u,v,\alpha) \in C^k(D')$, $\boldsymbol{r}_u \times \boldsymbol{r}_v \neq 0$; *then the unit normal to* S_α *has the form* $\boldsymbol{\tau} = (\boldsymbol{r}_u \times \boldsymbol{r}_v)/|\boldsymbol{r}_u \times \boldsymbol{r}_v|$. *Let the mapping* $(u,v,\alpha) \to (x,y,z)$ *be one-to-one in* $I \times D'$, *so that there is an inverse mapping* $u = u(x,y,z)$, $v = v(x,y,z)$, $\alpha = F(x,y,z)$ *with the definition domain* D. *Then, for the mean curvature* H *for* $k = 2$ *and for the Gaussian curvature* K *for* $k = 3$ *of the surface* S_α *formulas* (5)–(7) *hold. The quantities* $\mathrm{div}\,\boldsymbol{\tau}$, $\mathrm{div}\,\boldsymbol{S}(\boldsymbol{\tau})$, $\mathrm{div}\,\boldsymbol{T}(F)$ *in it are given by the formulas of Lemma 2 with the replacement of* u *by* F.

Let us obtain similar formulas for a family of surfaces given by some general properties, which are formulated below.

Let $\{S_\tau\}$ be a family of the surfaces S_τ with the unit normal $\boldsymbol{\tau} = \boldsymbol{\tau}(x,y,z)$, continuously filling the domain D in the space x, y, z. In this case, we will represent the principal direction by the unit vector $\boldsymbol{l}_i$ $(i = 1, 2)$ with the corresponding direction, and the vector $\boldsymbol{l}_i$ is the unit tangent vector line of the curvature L_i on S_τ. The vector $\boldsymbol{l}_i$ at the point $(x,y,z) \in S_\tau$ is equal to the derivative of the radius vector $\boldsymbol{r} = \boldsymbol{r}(x,y,z)$ of a point on the surface S_τ along the principal direction at the point (x,y,z). Let:

(A) one and only one surface $S_\tau \in \{S_\tau\}$ passes through each point $(x,y,z) \in D$;

(B) at each point $(x, y, z) \in D$, there exists the right system of the mutually orthogonal unit vectors $\boldsymbol{\tau}$, $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, where $\boldsymbol{\tau}$ is the unit normal, $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are the principal directions on the surface S_τ, passing through this point. For meeting this condition it is sufficient that each surface $S_\tau \in \{S_\tau\}$ be C^2-regular [12]. At each point $(x, y, z) \in S_\tau$ of flattening or rounding we can take any two mutually orthogonal directions, tangent to S_τ as $\boldsymbol{l}_1$, $\boldsymbol{l}_2$ [9–12]. Thus, three mutually orthogonal vector fields of the unit vectors $\boldsymbol{\tau}(x, y, z)$, $\boldsymbol{l}_1(x, y, z)$, $\boldsymbol{l}_2(x, y, z)$ are defined in the domain D;

(C) $\boldsymbol{\tau} \in C^n(D)$ (below $n = 1$ or 2), $\boldsymbol{l}_i \in C^1(D)$, $i = 1, 2$.

Remark 1. The properties (A)–(C) are valid for the above cases under the conditions of Theorem 1 and Corollary 2 (the role of S_τ is played by S_α).

Lemma 4. *Let the family $\{S_\tau\}$ of the surfaces S_τ with the unit normal $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z)$ satisfy conditions (A)–(C) for $n = 1$ in D. Then*

$$(\boldsymbol{l}_1 \cdot \operatorname{rot} \boldsymbol{l}_1) = (\boldsymbol{l}_2 \cdot \operatorname{rot} \boldsymbol{l}_2), \tag{8}$$

$$(\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\tau}) = 0 \quad \textit{in } D, \tag{9}$$

$\boldsymbol{\tau} = \operatorname{grad} u / |\operatorname{grad} u|$ *and the surface S_τ can be represented as $u(x, y, z) =$ const, where u is a scalar function.*

Proof. Using the general formulas [8 , § 17] $\operatorname{rot}[\boldsymbol{a} \times \boldsymbol{b}] = (\boldsymbol{b} \cdot \nabla)\boldsymbol{a} - (\boldsymbol{a} \cdot \nabla)\boldsymbol{b} + \boldsymbol{a} \operatorname{div} \boldsymbol{b} - \boldsymbol{b} \operatorname{div} \boldsymbol{a}$, $\operatorname{grad}(\boldsymbol{a} \cdot \boldsymbol{b}) = (\boldsymbol{b} \cdot \nabla)\boldsymbol{a} + (\boldsymbol{a} \cdot \nabla)\boldsymbol{b} + [\boldsymbol{b} \times \operatorname{rot} \boldsymbol{a}] + [\boldsymbol{a} \times \operatorname{rot} \boldsymbol{b}]$, taking into account the equalities $\boldsymbol{l}_1 = \boldsymbol{l}_2 \times \boldsymbol{\tau}$, $\boldsymbol{l}_2 = \boldsymbol{\tau} \times \boldsymbol{l}_1$, $(\boldsymbol{\tau} \cdot \boldsymbol{l}_1) = 0$, $(\boldsymbol{\tau} \cdot \boldsymbol{l}_2) = 0$ and the Rodriguez formula [6–12], written in the form $(\boldsymbol{l}_1 \cdot \nabla)\boldsymbol{\tau} = -k_1 \boldsymbol{l}_1$, $(\boldsymbol{l}_2 \cdot \nabla)\boldsymbol{\tau} = -k_2 \boldsymbol{l}_2$, for the two cases $\boldsymbol{a} = \boldsymbol{l}_2$, $\boldsymbol{b} = \boldsymbol{\tau}$ and $\boldsymbol{a} = \boldsymbol{\tau}$, $\boldsymbol{b} = \boldsymbol{l}_1$, we obtain $\operatorname{rot} \boldsymbol{l}_1 = (2k_2 + \operatorname{div} \boldsymbol{\tau})\boldsymbol{l}_2 - \boldsymbol{\tau} \operatorname{div} \boldsymbol{l}_2 + \operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{l}_2 + \operatorname{rot} \boldsymbol{l}_2 \times \boldsymbol{\tau}$, $\operatorname{rot} \boldsymbol{l}_2 = -(2k_1 + \operatorname{div} \boldsymbol{\tau})\boldsymbol{l}_1 + \boldsymbol{\tau} \operatorname{div} \boldsymbol{l}_1 + \boldsymbol{l}_1 \times \operatorname{rot} \boldsymbol{\tau} + \boldsymbol{\tau} \times \operatorname{rot} \boldsymbol{l}_1$. Multiplying scalarly the latter equalities, respectively, by $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$, we obtain the equalities $(\operatorname{rot} \boldsymbol{l}_1 \cdot \boldsymbol{l}_1) = (\operatorname{rot} \boldsymbol{l}_2 \cdot \boldsymbol{l}_2) - (\operatorname{rot} \boldsymbol{\tau} \cdot \boldsymbol{\tau})$ and $(\operatorname{rot} \boldsymbol{l}_2 \cdot \boldsymbol{l}_2) = (\operatorname{rot} \boldsymbol{l}_1 \cdot \boldsymbol{l}_1) - (\operatorname{rot} \boldsymbol{\tau} \cdot \boldsymbol{\tau})$. After adding and subtracting them we obtain (8), (9). □

Now we apply to the obtained equality $(\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\tau}) = 0$ the following theorem from Task 136 in [8, §17, P. 199]: the necessary and sufficient condition for a variable vector $\boldsymbol{a}$ could be represented in the form $\boldsymbol{a} = \varphi \operatorname{grad} u$, where φ and u are variable scalar functions, consists in performing the equality $(\boldsymbol{a} \cdot \operatorname{rot} \boldsymbol{a}) = 0$. Hence, taking into consideration the identity $|\boldsymbol{\tau}| = 1$, we obtain $\boldsymbol{\tau} = \operatorname{grad} u / |\operatorname{grad} u|$, i.e., $\boldsymbol{\tau}$ is the unit normal of the surface $u(x, y, z) = \text{const}$.

Lemma 4 and Theorem 1 imply

Theorem 2. *Let the family* $\{S_\tau\}$ *of the surfaces* S_τ *with the unit normal* $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z)$ *in* D *satisfy the conditions* (A)–(C). *Then, at each point* $(x, y, z) \in D$, *the mean curvature* H *for* $n = 1$ *and the Gaussian curvature* K *for* $n = 2$ *of the surface* S_τ *passing through this point have the divergence representations* (5), (6). *Moreover, for the principal* (*normal*) *curvatures* k_1, k_2 *and the quantities* $\operatorname{rot}\boldsymbol{l}_1$, $\operatorname{rot}\boldsymbol{l}_2$ ($\boldsymbol{l}_1$ *and* $\boldsymbol{l}_2$ *are the principal directions*), *the following formulas are valid*:

$$k_1 = -(\operatorname{rot}\boldsymbol{l}_1 \cdot \boldsymbol{l}_2), \qquad k_2 = (\operatorname{rot}\boldsymbol{l}_2 \cdot \boldsymbol{l}_1), \tag{10}$$

$$\operatorname{rot}\boldsymbol{l}_1 = -\boldsymbol{\tau}\operatorname{div}\boldsymbol{l}_2 + (k_2 - k_1)\boldsymbol{l}_2 + (\operatorname{rot}\boldsymbol{\tau} \times \boldsymbol{l}_2) + (\operatorname{rot}\boldsymbol{l}_2 \times \boldsymbol{\tau}), \tag{11}$$

$$\operatorname{rot}\boldsymbol{l}_2 = \boldsymbol{\tau}\operatorname{div}\boldsymbol{l}_1 + (k_2 - k_1)\boldsymbol{l}_1 + \boldsymbol{l}_1 \times \operatorname{rot}\boldsymbol{\tau} + \boldsymbol{\tau} \times \operatorname{rot}\boldsymbol{l}_1. \tag{12}$$

Remark 2. From formulas (10), a brief and simple proof of the representation $H = -\frac{1}{2}\operatorname{div}\boldsymbol{\tau}$ can be obtained, which does not use Lemma 4 and Theorem 1 (in contrast to Theorem 2). In fact, $\operatorname{div}\boldsymbol{\tau} = \operatorname{div}(\boldsymbol{l}_1 \times \boldsymbol{l}_2) = (\operatorname{rot}\boldsymbol{l}_1 \cdot \boldsymbol{l}_2) - (\operatorname{rot}\boldsymbol{l}_2 \cdot \boldsymbol{l}_1) = -k_1 - k_2 = -2H$ (we use the general formula [8, § 17] $\operatorname{div}(\boldsymbol{a} \times \boldsymbol{b}) = (\operatorname{rot}\boldsymbol{a} \cdot \boldsymbol{b}) - (\operatorname{rot}\boldsymbol{b} \cdot \boldsymbol{a})$). The formula $\boldsymbol{K} = -\frac{1}{2}\operatorname{div}\boldsymbol{S}(\boldsymbol{\tau})$ can also be obtained by an other way with the extremal property of the principal directions.

2. The geometric meaning of the fields $\boldsymbol{S}(\boldsymbol{\tau})$ and $\boldsymbol{T}(\boldsymbol{v})$ (or $\boldsymbol{T}(u)$) is explained by

Theorem 3. *Let* $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z)$ *be the field of the unit vectors in* D; *the family* $\{L_\tau\}$ *of the vector lines* L_τ *of the field* $\boldsymbol{\tau}$ *and the family* $\{S_\tau\}$ *of the surfaces* S_τ *with the normal* $\boldsymbol{\tau}$ *are mutually orthogonal in* D. *For example,* $\boldsymbol{\tau}$ *is the direction field of a vector field* $\boldsymbol{v} = |\boldsymbol{v}|\boldsymbol{\tau}$, $|\boldsymbol{v}| \neq 0$ *in* D. *Let the conditions* (A)–(C) *hold for* $n = 1$ *in* D. *Then the field* $\boldsymbol{S}(\boldsymbol{\tau})$ *of the form* (2) *at any point* $(x, y, z) \in D$ *is the sum of the three curvature vectors:* $\boldsymbol{S}(\boldsymbol{\tau}) = \boldsymbol{K}_\tau + \boldsymbol{K}_{g1} + \boldsymbol{K}_{g2} = \boldsymbol{K}_\tau + 2H\boldsymbol{\tau}$. *Here* $\boldsymbol{K}_\tau = (\boldsymbol{\tau} \cdot \nabla)\boldsymbol{\tau} = \operatorname{rot}\boldsymbol{\tau} \times \boldsymbol{\tau}$ *is the curvature vector of the vector line* L_τ *of the field* $\boldsymbol{\tau}$ *at the point* (x, y, z); $\boldsymbol{K}_{g1} = k_{g1}\boldsymbol{\tau}$ *and* $\boldsymbol{K}_{g2} = k_{g2}\boldsymbol{\tau}$ *are the curvature vectors* (*at the same point*) *of the two geodesic lines with the curvatures* k_{g1} *and* k_{g2} *on the surface* S_τ, *passing through the point* $(x, y, z) \in S_\tau$ *in any two mutually orthogonal directions. According to identity* (1) *the same geometric meaning under the same conditions has the field* $\boldsymbol{T}(\boldsymbol{v})$ (*the field* $\boldsymbol{T}(u)$ *in the case of* $\boldsymbol{v} = \operatorname{grad} u$).

3. The following relationship between the properties of surfaces and the properties of the vector fields $\boldsymbol{\tau}$ and $\boldsymbol{S}(\boldsymbol{\tau})$ follows from Theorem 2.

Corollary 3. *With the conditions and notations of Theorem 2 for the surface of* S_τ *be minimal* ($H = 0$ *on* S_τ) *or developable* ($K = 0$ *on* S_τ), *it is necessary and sufficient that the condition* $\operatorname{div}\boldsymbol{\tau} = 0$ *or* $\operatorname{div}\boldsymbol{S}(\boldsymbol{\tau}) = 0$, *respectively, should hold on* S_τ*. For all the surfaces* $S_\tau \in \{S_\tau\}$ *be minimal (developable), it is necessary and sufficient that the field* $\boldsymbol{\tau}$ *(the field* $\boldsymbol{S}(\boldsymbol{\tau})$*) be solenoidal in* D: $\operatorname{div} \boldsymbol{\tau} = 0$ $(\operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) = 0)$*. The sign of the mean curvature* H *(the Gaussian curvature* K*) at each point of the surface is opposite to that of the value* $\operatorname{div} \boldsymbol{\tau}$ *(the value of* $\operatorname{div} \boldsymbol{S}(\boldsymbol{\tau})$*) at this point.*

From Theorem 2 and the Ostrogradskii–Gauss formula, we obtain for integrals of H and K

Corollary 4. *With the conditions and notations of Theorem 2*

$$\iiint\limits_D H\, dx\, dy\, dz = \frac{1}{2} \iint\limits_{S_D} (\boldsymbol{\tau} \cdot \boldsymbol{\eta})\, dS,$$
$$\iiint\limits_D K\, dx\, dy\, dz = \frac{1}{2} \iint\limits_{S_D} (\boldsymbol{S}(\boldsymbol{\tau}) \cdot \boldsymbol{\eta})\, dS,$$

where S_D *is the piecewise smooth boundary of* D*,* dS *is its element,* $\boldsymbol{\eta}$ *is the inner unit normal to* S_D*. If* $S_D \in \{S_\tau\}$*, then (for* $\boldsymbol{\eta} = \boldsymbol{\tau}$*)*

$$\iiint\limits_D H\, dx\, dy\, dz = \frac{1}{2} S, \qquad \iiint\limits_D K\, dx\, dy\, dz = \frac{1}{2} \iint\limits_{S_D} H\, dS,$$

where S *is the area of the boundary* S_D*.*

Remark 3. By virtue of Remark 1 the assertions of Theorems 2, 3 and Corollaries 2–4 are valid for the family $\{S_\alpha\}$ under the conditions of Theorem 1 or Corollary 2 (with replacing S_τ by S_α).

Let us reformulate Theorems 1, 2 in terms of the vector field $\boldsymbol{v}$.

Theorem 4. *Let* $\boldsymbol{v} = \boldsymbol{v}(x, y, z) = |\boldsymbol{v}|\boldsymbol{\tau}$ *be the vector field with the modulus* $|\boldsymbol{v}| \neq 0$ *and the direction* $\boldsymbol{\tau}$ *in* D*. Assume that the family* $\{S_\tau\}$ *of the surfaces* S_τ *with the normal* $\boldsymbol{\tau}$*, which are orthogonal to the vector lines* L_τ *of the field* $\boldsymbol{v}$*, satisfy the conditions of Theorem 2. Then from the condition* $\boldsymbol{v} \in C^k(D)$ *follows* $\boldsymbol{\tau} \in C^k(D)$*. Then the identities* $H = -\operatorname{div}\{\boldsymbol{v}/|\boldsymbol{v}|\}/2$*,* $K = -\operatorname{div} \boldsymbol{T}(\boldsymbol{v})/2$ *hold in* D*, where* H *and* K *are the mean and the Gaussian curvatures of the surface* S_τ *at the point* $(x, y, z) \in D$*, the field* $\boldsymbol{T}(\boldsymbol{v})$ *being defined in Lemma 1. In this case,* $(\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\tau}) = 0 \Leftrightarrow (\boldsymbol{v} \cdot \operatorname{rot} \boldsymbol{v}) = 0 \Leftrightarrow \boldsymbol{v} = \varphi \operatorname{grad} u$*, where* φ *and* u *are some scalar functions. In the case of a potential field* $\boldsymbol{v} = \operatorname{grad} u$*, we have* $\boldsymbol{T}(\boldsymbol{v}) = \boldsymbol{T}(u)$*, where the field* $\boldsymbol{T}(u)$ *is defined in Lemma 2.*

4. Let us apply these general geometric formulas to the equations of mathematical physics for calculating the mean and the Gaussian curvature of the surfaces, which are defined by the solutions to these equations, in terms of these solutions.

Corollary 5. *Let* $\tau = \tau(x, y, z)$ *be the solution* (*the time field*) *of the eikonal equation* $\tau_x^2 + \tau_y^2 + \tau_z^2 = n^2(x, y, z)$ *in* D, $\tau \in C^3(D)$, *the refractive index* $n \in C^2(D)$. *In Theorems 1, 2, 4 we have* $\boldsymbol{v} = \operatorname{grad} \tau$, *the role of the surfaces* S_α *and* S_τ *is played by the wave fronts* $\tau(x, y, z) = \mathrm{const}$, *the role of the curves* L_τ *be the rays* (*the vector lines of the field* $\operatorname{grad}\tau$). *Then for the mean curvature* H *and the Gaussian curvature* K *of the front* S_τ *we have in* D: $H = -\operatorname{div}\{\operatorname{grad}\tau/n\}/2$, $K = -\operatorname{div} \boldsymbol{T}(\tau)/2$, *where* $\boldsymbol{T}(\tau) = \operatorname{grad} \ln n - \Delta\tau \operatorname{grad}\tau/n^2$.

Corollary 6. *Let* $u = u(x, y, z)$ *be the solution of Poisson equation* $\Delta u = -4\pi\rho(x, y, z)$ *in* D, *the potential* $u \in C^3(D)$, *the density* $\rho \in C^1(D)$. *In Theorems 1, 2 or 4 the role of the surfaces* S_α *or* S_τ *is played by the equipotential surfaces* $u(x, y, z) = \mathrm{const}$, *the role of the curves* L_τ *be the vector lines* (*lines of force*) *of the field* $\boldsymbol{v} = \operatorname{grad} u$. *Then for the mean curvature* H *and the Gaussian curvature* K *of the equipotential surfaces* S_τ *we have in* D: $H = -\operatorname{div}\{\operatorname{grad} u/|\operatorname{grad} u|\}/2$, $K = -\operatorname{div} \boldsymbol{T}(u)/2$, $\boldsymbol{T}(u) = \operatorname{grad} \ln|\operatorname{grad} u| + 4\pi\rho \operatorname{grad} u/|\operatorname{grad} u|^2$.

Corollary 7. *Let* $\boldsymbol{v} = \boldsymbol{v}(x, y, z) = v\boldsymbol{\tau}$ *be the velocity in Euler's hydrodynamic equations* $\boldsymbol{v}_t + \operatorname{grad} v^2/2 - \boldsymbol{v} \times \operatorname{rot} \boldsymbol{v} = \boldsymbol{F} - \operatorname{grad} p/\rho$, *which can be written as* $\boldsymbol{G} = -\boldsymbol{T}(\boldsymbol{v})$ *(*$= -\boldsymbol{S}(\boldsymbol{\tau})$*), where* $\boldsymbol{G} \overset{\mathrm{def}}{=} \{\boldsymbol{v}_t + \boldsymbol{v} \operatorname{div} \boldsymbol{v} + \operatorname{grad} p/\rho - \boldsymbol{F}\}/v^2$ *in* D; $v \overset{\mathrm{def}}{=} |\boldsymbol{v}|$, $\boldsymbol{v} \in C^2(D)$, *the pressure* $p \in C^2(D)$, *the density* $\rho \in C^1(D)$, *body force per unit of mass* $\boldsymbol{F} \in C^1(D)$. *Then for the mean curvature* H *and the Gaussian curvature* K *of the surfaces* S_τ *orthogonal to the streamlines* L_τ (*vector lines of the field* $\boldsymbol{v}$ *at* $t = \mathrm{const}$), *we have in* D: $H = -\operatorname{div}\{\boldsymbol{v}/v\}/2$, $K = \operatorname{div} \boldsymbol{G}/2$.

5. The formulas obtained in Theorems 1–4 allow us to give a simple geometric interpretation of the conservation laws derived in [2–4].

Corollary 8. *The conservation law* $\operatorname{div} \boldsymbol{S}^* = 0 \Leftrightarrow \operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) = 0$ *for a family* $\{L_\tau\}$ *of the plane curves* L_τ *on the plane* x, y, *where* $\boldsymbol{S}^*$ *is the sum of the curvature vectors of the curves* L_τ *and the curves* L_ν, *orthogonal to them, which is obtained in* [4], *is equivalent to the vanishing Gaussian curvature* K *of the cylindrical surfaces with the directive curves* L_ν *and the generating lines orthogonal to the plane* x, y.

Corollary 9. *The geometric meaning of the law of conservation* $\operatorname{div} \boldsymbol{T}(\tau) = 0$ *obtained in [3] for the time field* $\tau = \tau(x, y)$ *(for the solutions of the eikonal equation* $\tau_x^2 + \tau_y^2 = n^2(x, y)$*), where* $\boldsymbol{T}(\tau) = \boldsymbol{S}^*$ *is the sum of the curvature vectors of the rays* L_τ *and of the fronts* L_ν*, orthogonal to them, is that the Gaussian curvature* K *of the cylindrical surfaces with the directive curves* L_ν *and the generating lines orthogonal to the plane* x, y*, is zero. The geometric meaning of the conservation law* $\operatorname{div} \boldsymbol{G} = 0$ *obtained in [2] in the plane case for solutions of Euler's hydrodynamic equations, where the field* $(-\boldsymbol{G}) = \boldsymbol{S}^*$ *is the vector sum of the curvature vectors of the stream lines* L_τ *and the curves* L_ν*, orthogonal to them, is that the Gaussian curvature* K *of the cylindrical surfaces* S_τ *with the direction lines* L_ν *and the generating lines orthogonal to the plane* x, y *vanishes.*

References

[1] Megrabov A.G. Differential identities relating the Laplacian, the modulus of gradient and the gradient directional angle of a scalar function // Dokl. Math. — 2009. — Vol. 79, No. 1. — P. 136–140. [Dokl. Acad. Nauk. — 2009. — Vol. 424, No. 5. — P. 599–603].

[2] Megrabov A.G. Differential identities relating the modulus and direction of a vector field, and Euler's hydrodynamic equations // Dokl. Math. — 2010. — Vol. 82, No. 1. — P. 625–629. [Dokl. Acad. Nauk. — 2010. — Vol. 433, No. 3. — P. 309–313].

[3] Megrabov A.G. Some differential identities and their applications to the eikonal equation // Dokl. Math. — 2010. — Vol. 82, No. 1. — P. 638–642. [Dokl. Acad. Nauk. — 2010. — Vol. 433, No. 4. — P. 461–465].

[4] Megrabov A.G. Divergence formulas (conservation laws) in the differential geometry of plane curves and their applications // Dokl. Math. — 2011. — Vol. 84, No. 3. — P. 857–861. [Dokl. Acad. Nauk. — 2011. — Vol. 441, No. 3. — P. 313–317].

[5] Megrabov A.G. Differential identities relating and separating the modulus and direction of a vector field // Search of Mathematical Regularities of the Universe: Physical Ideas, Approaches, Concepts / Ed. by M.M.Lavrent'ev, V.N.Samoilov. — Novosibirsk: Academic publishing house "Geo", 2010. — P. 155–190. — (The selected transactions of VII Siberian conference on mathematical problems of physics of space-time of the complicated systems, devoted to the 100 anniversary of report G. Minkovsky "Space and time" (PST–2008), Novosibirsk, 21–24 September, 2008; Issue 7) (In Russian).

[6] Blaschke W. Differential Geometry. — Berlin: Springer, 1930. — Vol. 1 (In Russian: Elementary Differential Geometry. — Moscow–Leningrad: ONTI, 1935).

[7] Dubrovin B.A., Novikov S.P., Fomenko A.T. Modern Geometry: Methods and Applications. — Moscow: Nauka, 1979 (New York: Springer, 1984, 1985, 1990).

[8] Kochin N.E. Vectorial Calculus and the Fundamentals of Tensor Calculus. — Leningrad: GONTI, 1938 (In Russian).

[9] Finikov S.P. Differential Geometry Course. — Moscow: GITTL, 1952 (In Russian).

[10] Vygodsky M.Ya. Differential Geometry. — Moscow-Leningrad: ONTI, 1949 (In Russian).

[11] Novikov S.P, Taimanov I.A. Modern Geometric Structures and Fields. — Moscow: MCNMO, 2005 (In Russian).

[12] Poznyak E.G., Shikin E.V. Differential Geometry. The First Acquaintance. — Moscow: Moscow State University, 1990 (In Russian).