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A method for determination of wave rays in non-homogeneous media*

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Abstract. In non-homogeneous media with a variable wave propagation velocity it is not so easy to solve the boundary wave ray problem, i.e., to define a wave-ray, which connects two given points of the area. In this paper, an effective method for the wave-rays definition is proposed. The method does not exploit the differential equations of the wave rays. An approximate ray trace from the source point to any other grid-point of the computational area can be determined after solving (using the Huygence principle) only one direct problem for the travel-times definition.

1. Possible ways for determination of the wave rays

The wave rays are the fastest ways (traces) for the perturbation propagation from one point of a medium to another. In non-homogeneous media, these rays can be found by different ways. The first method includes the numerical solution to the wave ray differential equations [1]:

$$\frac{d\vec{x}}{dt} = \frac{\vec{p}}{n^2(\vec{x})}, \qquad \frac{d\vec{p}}{dt} = \nabla \ln n(\vec{x}), \tag{1}$$

where $\vec{p}$ is the ray direction vector, $n(\vec{x}) = 1/c(\vec{x})$, $c(\vec{x})$ is the wave propagation velocity at a point $\vec{x} = (x_1, ..., x_n)$, t is time. If we set the initial conditions

$$\vec{x}\big|_{t=0} = x^0, \qquad \vec{p}\big|_{t=0} = n(x^0) \cdot \vec{\nu}^0, \tag{2}$$

where $\vec{\nu}^0$ is the unit vector that defines the initial wave ray outgoing the direction from the point x^0. Having solved the initial value problem (1), (2), using, for example, the Runge–Kute method, it appears possible to find the further wave ray trace as a sequence of points on the plane surface (in the two-dimensional case).

Using such an approach, the wave ray which connects two fixed points of an area can only be found within one computational experiment only by a lucky choice of the initial outgoing direction of the ray. Usually, it is necessary to build a lot of wave rays before one of them comes to the neighborhood of the receiving point. In a three-dimensional case with a significant non-homogeneity, such a way for solving the boundary problem

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is too long and not effective. It can give a result only with the help of some special optimizing tricks.

Another way for definition of the wave rays is based on the kinematics of the wave front. In this attempt, the initial wave front is regarded as the number of computational points along the circle around the source point. Then each point is moving toward the normal (to the front line) direction. The moving rate is equal to the wave propagation velocity at each computational point. After each time step, every point of the front moves for a distance, which is determined by this value. The normal direction is defined using the location of two neighboring points of the wave front. As a result of such a computational technology, we will obtain the approximate wave rays as a sequence of positions of one wave front point. If we draw the line through all the defined wave front points, which are corresponding to the same time moment, then we will obtain the wave front positions at different moments of time. If, in addition, we draw the traces of each computational point, then all the area will be divided into small segments (Figure 1). The receiving point is located in one of these segments. So, the wave ray that connects the source and the receiver is situated between the ones, which give the boundaries of the segment in question.

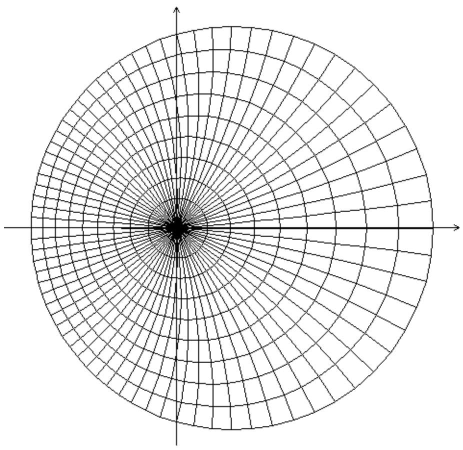

Figure 1. Wave rays and tsunami isochrones from the round source above the parabolic bottom

As an example, in Figure 1, the wave rays from a small round source are shown for the model bottom relief. In this rectangular area, the depth is increasing from a minimum value on the left boundary according to the formula

$$H(x, y) = ax^2, \tag{3}$$

where x is a distance from the left boundary, a is a constant coefficient. The initial wave front includes 72 computational points which are located in equal intervals along the circle of 10 km radius (see Figure 1). The propagation velocity was defined by the Lagrange formula

$$c = \sqrt{gH}, \tag{4}$$

where g is the gravity acceleration, H is the value of the depth. The propagation direction of each computational point is defined as orthogonal to the segment that connects from both sides two neighboring points.

2. Travel-time computations in non-homogeneous areas

For travel-time computations in areas with a complicated topology (islands and narrow straits), the methods based on the Huygence principle are more effective [3]. The essence of this principle is, in that all points of the water area, where the wave perturbation has already arrived, become the sources of the wave radiation and the radiate wave energy in all directions. Thus, the algorithm of computation is based on looking through all the neighboring to a wave-front grid-points (in which perturbation has not arrived yet) and calculation of travel-times there, as minimum among all possible travel-times from the nearest points. The sum of travel-times from the initial source up to each wave-front point and from the points to the regarded grid-point is minimized among all possible travel ways. Let us explain this by Figure 2. Let the travel-times calculations be carried out in the area with a rectangular computational grid. This means that at all points of this grid the values of depths are known, and it is required to find the tsunami travel-times from a specified source (one or several grid-points) up to all the other grid-points.

Let us schematically represent in Figure 2 a fragment of the computational area. Here the black small squares designate those points of the grid, in which the perturbation from the initial tsunami source at this time moment has already came, the travel-times at these grid-points being known. We need to find the travel-time from the source up to the point A. Relative to the point A, the neighboring points, where the travel-times are known, there will be the points B, C, D, and E. Let the travel-times up to them be equal to T_B, T_C, T_D, and T_E, respectively. Between the adjacent grid-points the depth is varying according to the linear law. Let us find a tsunami travel-time between two neighboring grid-points. The distance between them is equal to L, and the depth varies from the value H_1 up to H_2. Let us introduce an auxiliary value α — the angle of declination of bottom, $\operatorname{tg}\alpha = (H_2 - H_1)/L$. Then the travel-time will be expressed as

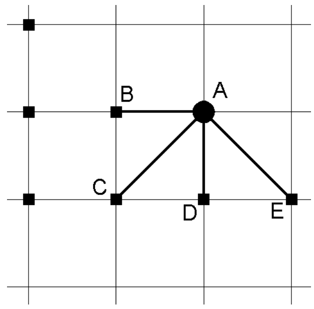

Figure 2. The scheme of determining travel-times using the Huygence principle

$$T = \int_0^L \frac{dl}{\sqrt{g(H_1 + l\operatorname{tg}\alpha)}} = \frac{1}{\sqrt{g\operatorname{tg}\alpha}}\int_0^L \left(l + \frac{H_1}{\operatorname{tg}\alpha}\right)^{-1/2} d\left(l + \frac{H_1}{\operatorname{tg}\alpha}\right)$$
$$= \frac{2}{\sqrt{g\operatorname{tg}\alpha}}\left(l + \frac{H_1}{\operatorname{tg}\alpha}\right)^{1/2}\Bigg|_0^L = \frac{2}{\sqrt{g\operatorname{tg}\alpha}}\cdot\frac{\sqrt{H_2}-\sqrt{H_1}}{\sqrt{\operatorname{tg}\alpha}}$$
$$= \frac{2}{\sqrt{g}\operatorname{tg}\alpha}\cdot\frac{H_2 - H_1}{\sqrt{H_2}+\sqrt{H_1}} = \frac{2L}{\sqrt{gH_2}+\sqrt{gH_1}}. \tag{5}$$

Therefore, the tsunami travel-time between the neighboring grid-points is equal to a distance between them divided by the arithmetically averaged velocity of a tsunami at these grid-points. Thus, in order to find the travel-time from a source up to a point A, it is necessary to find a minimum of four time values

$$\begin{aligned} T_1 &= T_B + \frac{2\Delta x}{\sqrt{gH_B} + \sqrt{gH_A}}, \quad T_2 = T_C + \frac{2\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\sqrt{gH_C} + \sqrt{gH_A}}, \\ T_3 &= T_D + \frac{2\Delta y}{\sqrt{gH_D} + \sqrt{gH_A}}, \quad T_4 = T_E + \frac{2\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\sqrt{gH_E} + \sqrt{gH_A}}, \end{aligned} \tag{6}$$

where Δx and Δy are steps of the grid in horizontal and vertical directions and H_A, H_B, H_C, H_D, and H_E — the depth values at the corresponding points. A minimum of values T_i $(i = 1, 2, 3, 4)$ will give us the tsunami travel-time from the source up to the point A. In such a way, it is possible to find point by point the travel-times to all the points of a computational grid.

Let us speak about the meaning of neighboring points. Such grid-points relative to the considered one are determined by the so-called templates. The simplest one is the eight-dot template (Figure 3).

In such a template, the following eight grid-points: $(i-1, j+1)$, $(i, j+1)$, $(i+1, j+1)$, $(i+1, j)$, $(i+1, j-1)$, $(i, j-1)$, $(i-1, j-1)$, $(i-1, j)$ will be the neighboring ones to the point A with the grid coordinates (i, j). The travel-times from each of them to the central point are calculated along eight straight rays (segments). Therefore, this template is sometimes called eight-radial. However, this template is too simplified, and as a result, the wave front computations from a point source using this template give above the bottom of constant depth the octagonal front line instead of a circle.

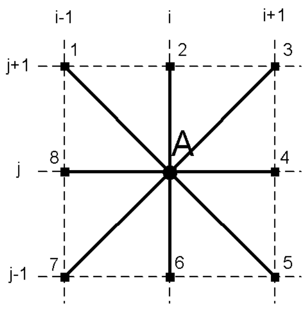

Figure 3. The eight-dot template for computation of tsunami isochrones

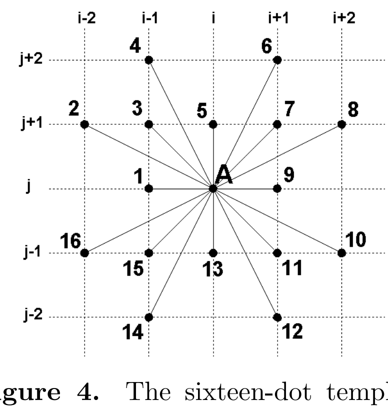

Figure 4. The sixteen-dot template for tsunami travel-time calculations on a rectangular computational grid

More precise results can be obtained if for computations we use a sixteen-dot template (Figure 4).

In this template, the neighbor to the point A, that has the grid coordinates (i, j), are the grid-points whose indices differ from the grid coordinates of the point A not only by one, but also by some grid-points, whose coordinates differ by two (see Figure 4). In this case if at any of these 16 points the tsunami travel-time is already known, it is possible to find the travel-time to the point A (the center of the template).

3. Method for solving the boundary problem for the wave rays

However, this method for travel-time computations is not able to give directly the traces of the wave rays. In order to define the wave ray traces within such an attempt, the following algorithm can be proposed. If we need to find the ray between two given points of the computational domain, then we must calculate the travel-times from one of these points to all the other grid points of the computational domain. In the travel-time definition at the point (i, j), we input to the computer memory the grid coordinates of the neighboring point, which is used for calculating the travel-time at the point (i, j). After finishing the computations, two matrices with the dimensions of the whole computational area will be filled. The entries of the first array are the horizontal coordinates, and in the second array there are vertical grid coordinates of those points. Using these two arrays it is very easy to reconstruct a ray trace from an arbitrary grid-point to the source point (starting with a receiving point).

Figure 5 shows an example of the ray reconstruction process in a small sub-area of the whole computational domain. In the top and the left sides of the picture, the grid coordinates are indicated. It is needed, for example, to build a wave ray between the points with grid coordinates $(10, 10)$ and $(16, 11)$. At the first step, we calculate travel times in the whole computational domain from the point source, which is installed at the point $(10, 10)$. For some grid-points in Figure 5 in parentheses we give pairs of numbers from those two arrays, which were obtained during travel-time computations. If we begin with the point $(16, 11)$, the sequence of grid-points, which gives the quickest way for wave propagation from point $(10, 10)$ to the point $(16, 11)$, can be easily defined $((16, 11), (14, 12), (13, 12), (11, 11), (10, 10))$. If we draw a line through these points, then we will obtain an approximate ray trace connecting points $(10, 10)$ and $(16, 11)$.

The wave rays, defined by this method, consist of the segments, being the rays of the template ("star") (see Figures 3 and 4). When we use an 8-dot template (see Figure 3), segments of the defined ray can have only eight variants of direction. When using a 16-dot template, the number of

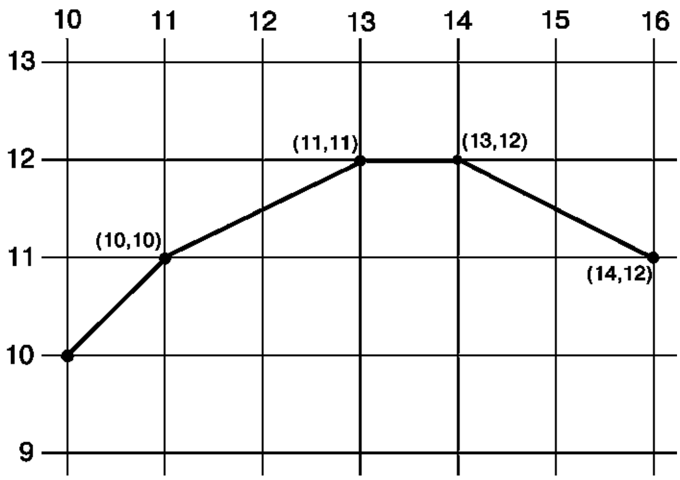

Figure 5. Reconstruction of the ray trace between two points

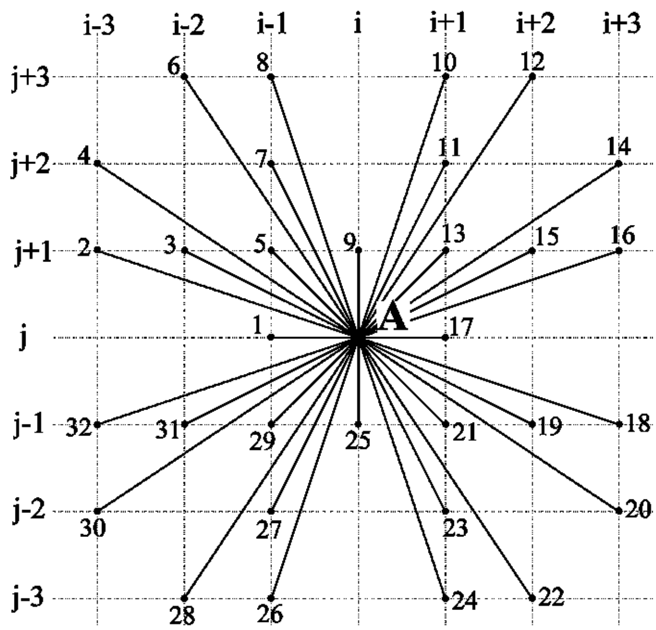

Figure 6. The 32-dot template for travel-time computations

variants increases up to 16. It means that approximation of a real wave ray is significantly improved. In the areas, where depth is varying slowly, it is possible to calculate travel times by a 32-dot template (Figure 6). In this case, it is necessary to calculate travel-times along the long segments more accurately. Figure 5 shows that the directivity of the neighboring segments of which the ray consists can differ only by 10–15 degrees.

Figure 7. The wave ray above the uniform bottom slope built with the use of the proposed method

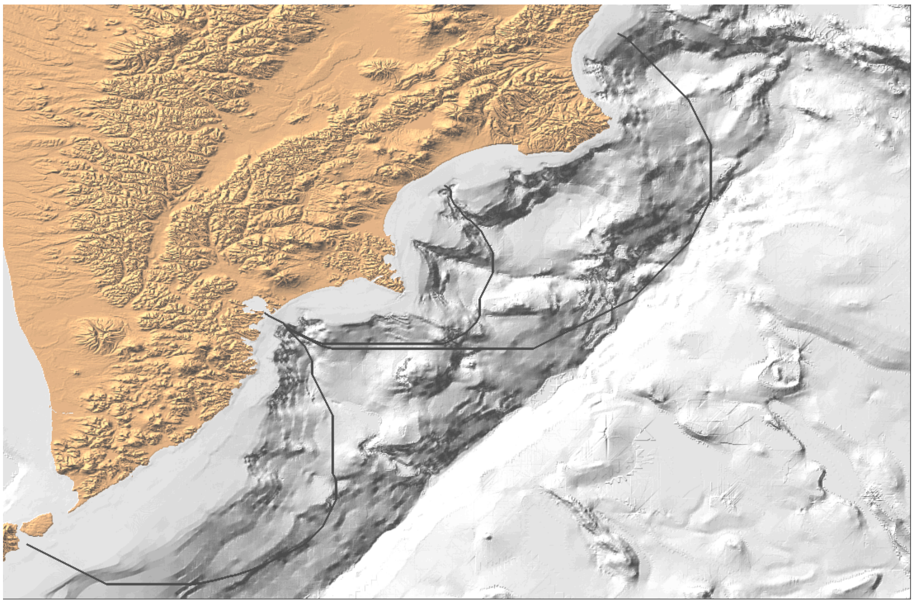

Figure 8. The bottom topography and the wave rays near Kamchatka

As an example, Figure 7 shows the reconstructed wave ray between two points above the uniform bottom slope. Here the depth linearly increases when going away from the lower boundary of the computational domain. The shape of the ray is in good correlation with the analytical solution, being a segment of the cycloid [2].

Let us consider a real area near the Kamchatka peninsular with a complicated bottom relief. We need to define the wave rays connecting the mouth of the Avacha harbor and three other points of this area. One point is a Severo-Kurilsk village and the other two points are located on the Kamchatka shelf (Figure 8). At the first step, we put a source at the entrance of the Avacha harbor and define the tsunami travel-times to all the grid points of this computational domain. Then using the described method we have defined the shown ray traces (see Figure 8).

As conclusion it should be noted that the method proposed can be effectively used in a three-dimensional case, for example, in 3D seismic problems.

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