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Some analytical solutions for tsunami wave rays and front*

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Abstract. Methods for computing wave kinematics are wide-spread for studying the wave propagation in non-homogeneous media. Exact analytical solutions for nontrivial media are needed for testing such methods. The formula describing the wave-front shape has been found for some media with a power dependence of conductivity on only one spatial coordinate. Also, the formulas for wave-ray traces over the linear and parabolic bottom profile were obtained. The results of this research are used for numerical methods testing.

1. Solution of the eikonal equation for some kinds of the cylindrical bottom relief

Analytical solutions for the tsunami wave-front behavior above an uneven bottom can be described by the so-called eikonal equation

$$\Big(\frac{\partial f}{\partial x}\Big)^2 + \Big(\frac{\partial f}{\partial y}\Big)^2 = \frac{1}{v^2(x,y)}, \tag{1}$$

where $v(x, y)$ is the wave propagation velocity distribution in the environment. The position of a wave front at the time is given by the equation

$$f(x, y) = C. \tag{2}$$

The equation $f(x, y) = 0$ sets a tsunami source boundary. As a matter of fact, solutions of equation (1) are known only for rather a limited number of functions v [1], therefore usually the wave front position is obtained by numerical methods.

In [2] it is shown, how, in an important case of power dependence v on a coordinate it is possible to find the front positions, without solving equation (1). Let us present a little different from that in [2], a description of this algorithm on the following equation as an example

$$f_x^2 + f_y^2 = (ky)^{2\alpha}, \tag{3}$$

where k, α are numbers. In these designations $v = (ky)^{-\alpha}$. Let us consider the case when the propagation velocity increases, going off the coast, i.e., $\alpha < 0$. Additional restriction for α will be formulated later.

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A solution of equation (3) is sought for as follows

$$f = y^{\alpha+1} p(z). \tag{4}$$

where

$$z = \frac{x}{y}. \tag{5}$$

Then $f_x = y^{\alpha} p'$, $f_y = y^{\alpha}((\alpha+1)p - zp')$, and after the substitution into (3) we will obtain

$$(p')^2 + \big((\alpha+1)p - zp'\big)^2 = k^{2\alpha}. \tag{6}$$

From the above equation follows that there is a function $u(z)$ such that

$$p' = k^{\alpha} \sin u, \qquad (\alpha+1)p - zp' = k^{\alpha} \cos u. \tag{7}$$

From here follows

$$(\alpha+1)p = k^{\alpha}(z \sin u + \cos u). \tag{8}$$

Differentiating both parts of (8), we obtain

$$(\alpha+1)p' = k^{\alpha}\Big(\sin u + z\cos u \frac{du}{dz} - \sin u \frac{du}{dz}\Big),$$

or, taking into account the first equation of the system (7),

$$\alpha \sin u = (z \cos u - \sin u)\frac{du}{dz},$$

we will rewrite this equation in the form

$$\frac{dz}{du} = \frac{1}{\alpha} \operatorname{ctg} u \cdot z - \frac{1}{\alpha}.$$

We have obtained the linear differential equation whose common solution can be expressed as

$$z = |\sin u|^{1/\alpha}\Big(C_1 - \frac{1}{\alpha}\int \frac{du}{t} |\sin u|^{1/\alpha}\Big).$$

So, the solution of equation (6) in the parametrical form looks like the following:

$$\begin{cases} p = \dfrac{1}{\alpha+1} k^{\alpha}(z \sin u + \cos u), \\ z = |\sin u|^{1/\alpha}\Big(C_1 - \dfrac{1}{\alpha}\displaystyle\int \frac{du}{|\sin u|^{1/\alpha}}\Big). \end{cases} \tag{9}$$

If we now substitute an expression for f from (4) into the equation of front (2) we will have $y^{\alpha+1}p = C$, and, substituting $p(u)$ from (9), we have

$$y(u) = \left(\frac{C}{p(u)}\right)^{\frac{1}{\alpha+1}}.$$

Taking into account (5), we can rewrite $x(u) = y(u)z(u)$. Finally, we obtain the equation of the wave front in the parametrical form:

$$x(u) = \left(\frac{C}{p(u)}\right)^{\frac{1}{\alpha+1}} z(u), \qquad y(u) = \left(\frac{C}{p(u)}\right)^{\frac{1}{\alpha+1}}, \tag{10}$$

where $p(u), z(u)$ can be obtained from (9) and $\alpha \neq -1$. The integral, entering in (9), is expressed through an elementary function if only $1/\alpha$ is an integer number. For these values, formulas (10) give the exact expression for the wave front. At other values α numerical integration is required. Figures 1–4 present wave front shapes set by equations (10) with different values of parameters.

Curves were constructed with the help of MathCad software. A range of the parameter u variations and the step Δ of its change, along with other parameters are given in figure captions. It is necessary to note that at $\alpha < -1$, the exponent becomes less than zero and with the growth of C, the front line "is pulled together" to the origin of coordinates, i.e., formulas (10) describe the wave propagation process in the "conversion" time. Therefore, for finding the wave front corresponding to the time C it is necessary to substitute $1/C$ into formulas (10).

In order to test some numerical methods, we will consider only the cases when $1/\alpha$ is integer and $-1 < \alpha < 0$. Here the integral included into (9) is expressed through elementary functions, and the parameter u is varying from $-\pi$ up to π. The parameter C_1 is equal to zero. In this

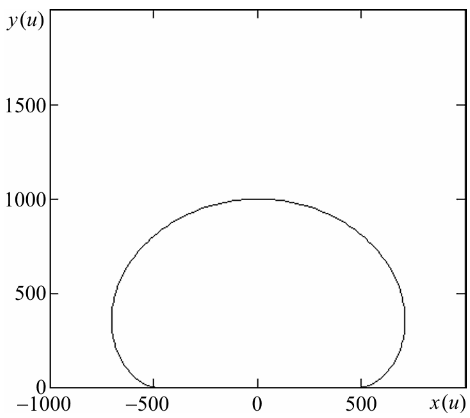

Figure 1. $\alpha = -\frac{1}{3}$, $C = 150$, $k = 1$, $C_1 = 0$, $-\pi \le u \le \pi$, $\Delta = \frac{\pi}{100}$

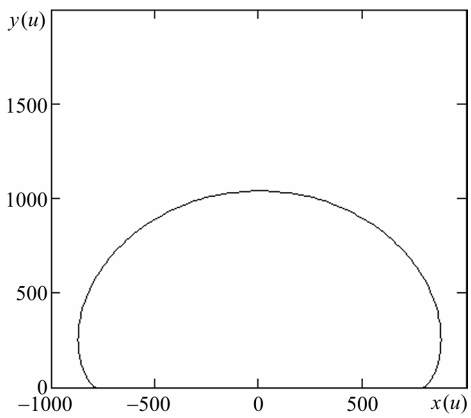

Figure 2. $\alpha = -\frac{1}{8}$, $C = 500$, $k = 1$, $C_1 = 0$, $-\pi \le u \le \pi$, $\Delta = \frac{\pi}{100}$

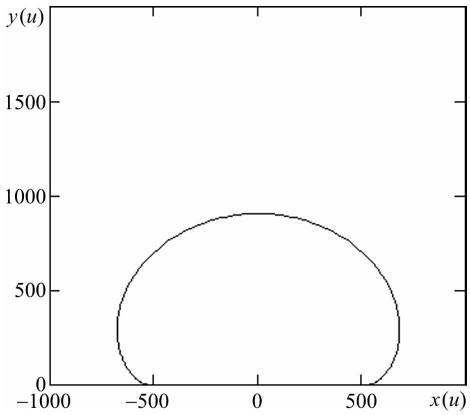

Figure 3. $\alpha = -\frac{1}{4}$, $C = 220$, $k = 1$, $C_1 = 0$, $-\pi \le u \le \pi$, $\Delta = \frac{\pi}{100}$

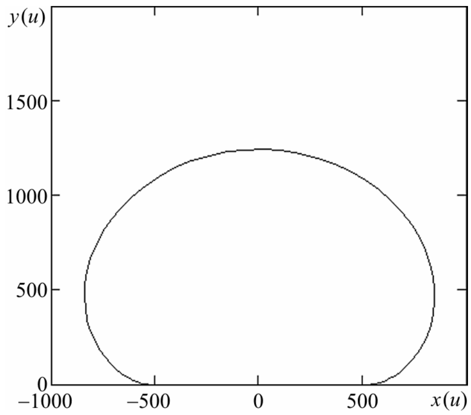

Figure 4. $\alpha = -\frac{2}{5}$, $C = 120$, $k = 1$, $C_1 = 0$, $-\pi \le u \le \pi$, $\Delta = \frac{\pi}{100}$

case, the front of the wave propagating from the source at the origin of coordinates can be found. In addition, with the help of these formulas, it is possible to gain inside into the new facts that are useful to apply. Let us give an example. Let the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ locate on isochrones of the function f being a solution to equation (3). Isochrones correspond to the times C_1 and C_2, respectively, the point P locating on the straight line OQ (Figure 5). From formula (10), we have

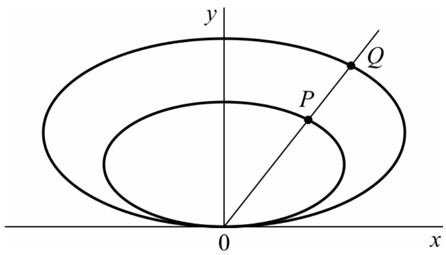

Figure 5

$$C_1 = y_1^{\alpha+1} p(z_1), \quad C_2 = y_2^{\alpha+1} p(z_2).$$

As the points P and Q locate on one straight line passing the origin of coordinates, then $z_1 = z_2$, and hence

$$\frac{C_1}{C_2} = \frac{y_1^{\alpha+1}}{y_2^{\alpha+1}}.$$

Taking into account the point that

$$\frac{y_1}{y_2} = \frac{OP}{OQ}$$

we finally have

$$C_2 = \Big(\frac{OQ}{OP}\Big)^{\frac{1}{\alpha+1}} \cdot C_1. \tag{11}$$

This formula allows building isochrones of the function f for any instant of time if at least one of such isochrones is known. Moreover, one isochrone allows us to find the tsunami arrival time at any point of an area. This fact can also be used for testing on numerical algorithms.

Formulas (10) are inapplicable when $\alpha = 0$ or $\alpha = -1$. In the first case, a medium is homogeneous and the front line represents circles having the center in the origin of coordinates. We will find the position of a front line from the source located at the point $(0, 0)$ for the second case. We search for a solution in the form $f = p(z) + a \ln y$, where $z = x/y$ will be used as before. Then

$$f_x = \frac{1}{y} p', \qquad f_y = \frac{1}{y}(a - zp').$$

After substitution of these formulas into (3), we will have

$$(p')^2 + (a - zp')^2 = \frac{1}{k^2}. \tag{12}$$

As well as earlier, let us assume

$$p' = \frac{1}{k} \sin u, \qquad a - zp' = \frac{1}{k} \cos u.$$

Hence $a = \frac{1}{k}(z \sin u + \cos u)$ and $z = \frac{ak - \cos u}{\sin u}$. Further we have

$$\frac{dz}{du} = \frac{1 - ak \cos u}{\sin^2 u}, \qquad \frac{dp}{du} = p' \frac{dz}{du} = \frac{1}{k} \cdot \frac{1 - ak \cos u}{\sin u}.$$

Then

$$p = \frac{1}{k} \int \frac{du}{\sin u} - a \int \frac{\cos u}{\sin u} du = \frac{1}{k} \ln \left| \operatorname{tg} \frac{u}{2} \right| - a \ln |\sin u|.$$

Thus, the solution of equation (12) in the parametrical form will be expressed as

$$p(u) = \frac{1}{k} \ln |\operatorname{tg} \frac{u}{2}| - a \ln |\sin u|, \qquad z(u) = \frac{ka - \cos u}{\sin u}.$$

These formulas give the solution of equation (12) for all the values of the parameter u when the right-hand sides of formulas are finite.

If $f = C$ is an equation of an isochrone, then $p + a \ln y = C$, and after certain transformations we arrive at

$$y(u) = e^{C/a} |\sin u| \cdot \left| \operatorname{tg} \frac{u}{2} \right|^{-1/ka}.$$

The corresponding abscise is defined from (5): $x(u) = y(u)z(u)$. Finally,

$$x(u) = e^{C/a} \operatorname{sgn}(\sin u) \cdot (ka - \cos u) \cdot \left| \operatorname{tg} \frac{u}{2} \right|^{-1/ka},$$
$$y(u) = e^{C/a} |\sin u| \cdot \left| \operatorname{tg} \frac{u}{2} \right|^{-1/ka}.$$

At $a = 1/k$ this system is reduced to the form

$$x(u) = e^{kC} \sin u, \qquad y(u) = e^{kC}(1 + \cos u).$$

In this case, the wave front represents a circle of radius e^{kC} with the center at the point $(0, e^{kC})$. For example, when $C = 0$, the front line will be the unit circle with the center at the point (0,1). If r_0 is the circle radius, then

$$r_0 = e^{kC_0} \quad \text{and} \quad C_0 = \frac{1}{k} \ln r_0.$$

If $r_0 < 1$, then C_0 is negative. When C_0 decreases, these formulas describe the process of wave propagation into the above-specified unit circle. If the time scale begins from the negative value $\frac{1}{k} \ln r_0$, i.e., the time is set by the expression

$$C = T + \frac{1}{k} \ln r_0, \qquad T \geq 0,$$

where T is new value of time, then we obtain the formulas, describing the wave propagation from r_0 radius circle with the center at the point $(0, r_0)$. From the formulas obtained it follows that the front line at the moment T presents the circle of radius $r_0 e^{kT}$ with the center at the point $(0, r_0 e^{kT})$. Isolines of this function are presented in Figure 6.

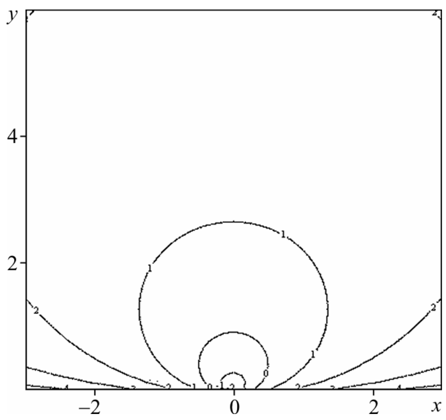

Figure 6

2. Exact analytical formulas for wave-ray traces above the sloping and parabolic bottom

The wave ray constructing problem is very important in the tsunami research, and all considerations in this section will be connected to the kinematics of tsunami waves. Now we will derive some exact mathematical formulas for the wave-ray traces above some types of a model bottom topography. The exact trajectory of a wave beam over an inclined bottom can be defined as well from the laws of geometrical optics. In particular, it is possible to use the Snell law for searching the refraction angle of a wave ray in a medium with a variable optical conductivity. According to this law if in the 2D wave-conducting medium the ray arrives at an angle α_1 (the direction between a wave ray and a normal to the border) to the rectilinear border where the conductivity rate (velocity of a signal propagation) varies from α_1 to α_2, then passing the border its direction α_2 will change according to the formula

$$\frac{\sin \alpha_1}{a_1} = \frac{\sin \alpha_2}{a_2}. \tag{13}$$

Thus, in the media, where the conductivity value a varies only along one spatial axis (for example, $a = a(y)$), the inclination of a wave ray associated with the direction of conductivity changing α varies by the formula

$$\frac{\sin(\alpha(y))}{a(y)} = \text{const}\,. \tag{14}$$

Assuming a sloping bottom, the optical conductivity (propagation velocity of tsunami waves) is defined by the Lagrangian formula $a = \sqrt{gH}$ (g is acceleration of gravity, H is water depth), which in the case of a bottom slope looks like

$$a(y) = \sqrt{gy \operatorname{tg} \beta}, \tag{15}$$

where β is an angle of the bottom slope. Hence, the relation between the direction of a ray (an optimum trajectory) and a distance to the coast will look like

$$\sin^2 \alpha = \text{const} \cdot y, \tag{16}$$

where the value of a constant is defined from a ray inclination at some distance to the coast (the axis OX). If we consider α as parameter on which y depends, then from (16) follows

$$dy = \frac{2 \sin \alpha \cos \alpha}{\text{const}} d\alpha. \tag{17}$$

As the value $(\pi/2 - \alpha)$ is an angle of a wave ray inclination (the function $y(x)$) graphics to the horizon, then, by definition of a derivative function of one variable, the following equality is valid:

$$\frac{dy}{dx} = \frac{\sin\left(\frac{\pi}{2} - \alpha\right)}{\cos\left(\frac{\pi}{2} - \alpha\right)} = \frac{\cos \alpha}{\sin \alpha},$$

or

$$dx = dy \frac{\sin \alpha}{\cos \alpha}, \qquad 0 \le \alpha \le \frac{\pi}{2}. \tag{18}$$

From (17) and (18) follows

$$dx = \frac{2 \sin \alpha \cos \alpha}{\text{const}} \frac{\sin \alpha}{\cos \alpha} d\alpha = \frac{2 \sin^2 \alpha}{\text{const}} d\alpha. \tag{19}$$

Thus, assuming x and y depend on the parameter $t = 2\alpha$ and using trigonometric formulas for sine and cosine of a double angle, we arrive from (17) and (19) to the following formulas:

$$dy = \frac{\sin t}{2 \cdot \mathrm{const}}\, dt, \qquad dx = \frac{1 - \cos t}{2 \cdot \mathrm{const}}\, dt. \tag{20}$$

After integration of equalities (20), the equations of a wave ray trajectory in the parametric form are obtained

$$x(t) = C_1(t - \sin t) + C_2, \quad y(t) = C_1(C_3 - \cos t), \qquad t \in [0, 2\pi].$$

This is a parametric form of the cycloid equation. Here, the constants C_2, C_3 are defined from the condition a cycloid movement through the point of the origin of coordinates. According to the Snell law (16), at the point $(0, 0)$, the parameter t is equal to zero. At $y = 0$, the angle α along with the parameter $t = 2\alpha$ become equal to zero. Therefore $C_2 = 0$, $C_3 = 1$. Finally, the equations of a wave ray in the parametric form will be written down as follows:

$$x(t) = C_1(t - \sin t), \quad y(t) = C_1(1 - \cos t), \qquad t \in [0, 2\pi]. \tag{21}$$

When presenting the equations in such a form, the parameter t is a doubled angle of a slope of a ray with respect to the normal to the coastal line, and the value of C_1 is defined for each concrete case. If the boundary value problem for a wave ray is solvable, the value of the parameter C_1 is defined from the condition of the ray through the point (x_1, y_1), thus the second point is the origin of coordinates. If the wave ray, which at a distance y_1 from the coast has been directed at an angle α_1 with respect to the normal movement of coastal line (the axis OX), equations (16) and (20) yield the required value

$$C_1 = \frac{y_1}{2 \sin \alpha_1}. \tag{22}$$

Thus, we have determined the equations describing a wave ray over a sloping bottom, proceeding the laws of movement of a ray in medium with a variable conductivity. Earlier in 1980 [3], the same equations were obtained owing to the fact that a wave ray is an optimal trajectory of wave signal propagation.

In addition, we will find an exact trajectory of a wave ray for one more type of a model bottom relief—parabolic. This means that a depth increases proportional to the square of a distance to the coast. In this case, the Lagrangian formula ($a = \sqrt{gH}$) for the wave propagation velocity will look like

$$a(y) = \sqrt{g b_1 y^2} = b_2 y, \tag{23}$$

where b_1 and b_2 are constants in all the considered coastal area. The Snell law (14) in this case gives the following formula for the direction of a ray with respect to the normal to the coastline

$$y(\alpha) = b_3 \sin \alpha, \tag{24}$$

where b_3 is a certain constant.

Let us consider the following problem. From the point $(0, y_0)$, being at a distance y_0 from the coast, a wave ray is emitted in parallel to the coastal line (Figure 7). At the source point, a wave ray has the direction $\alpha = \pi/2$. An angle between the wave ray $y(x)$ and the coastline $y = 0$ (see Figure 7) will be $\beta = \pi/2 - \alpha$. Hence, we have

$$\frac{dy}{dx} = -\operatorname{tg}\beta = -\frac{\sin\beta}{\cos\beta}, \quad 0 < \beta < \frac{\pi}{2},$$

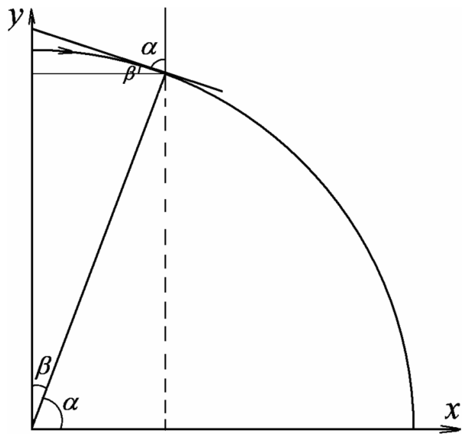

Figure 7

or

$$dx = -dy\frac{\cos\beta}{\sin\beta}. \tag{25}$$

From (24) and (25) follows

$$dx = b_3 \sin\beta \cdot d\beta \frac{\cos\beta}{\sin\beta} = b_3 \cos\beta \cdot d\beta.$$

Assuming that the angle β in the course of movement of a wave ray changes from zero to some value β_1, after integration we come to

$$x = b_3 \sin\beta \,\Big|_0^{\beta_1} = b_3 \sin\beta_1 = b_3 \cos\alpha_1, \tag{26}$$

where $\alpha_1 = \pi/2 - \beta_1$. At the same time, from (24) we have

$$y = b_3 \sin\alpha_1, \tag{27}$$

for any value α_1 in the range from $\pi/2$ to zero. Formulas (26), (27) represent a parametric notation of the equation of a circle with radius b_3. This radius is easily defined from formula (24). If a ray has initially been directed in parallel to the coastal line, a radius will be equal to the off-coast distance at this moment. If a ray inclination angle with respect to the normal to the coast α_0 and the off-coast distance y_0 at this moment are known, then the circle radius r containing the trajectory of this wave ray can be determined from formula (24)

$$r = b_3 = \frac{y_0}{\sin\alpha_0}. \tag{28}$$

If $\alpha_0 = 0$, a radius will infinitely big, and the ray trajectory representing a straight line orthogonally directed to the coast.

Unlike the case with a sloping bottom, the boundary value problem for the wave ray can be easily solved. Let two points be known (a source and a receiver) in the area with a parabolic bottom relief (23), thus the receiver is situated on the coastal line (the point of the origin of coordinates $(0, 0)$), the source being located at the point (x_0, y_0). Let for definiteness $y_0 > x_0 > 0$. This means that the wave ray monotonously approaches the coast (Figure 8).

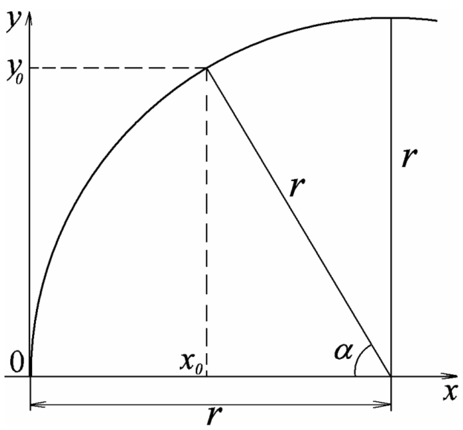

Figure 8

After transition to the parameters r and α (see Figure 8), the source coordinates will be expressed as

$$x_0 = r - r\cos\alpha, \quad y_0 = r\sin\alpha, \quad 0 < \alpha < \frac{\pi}{2}. \tag{29}$$

Further, the value of the angle α can be found from the coordinates ratio at the source-point:

$$\frac{x_0}{y_0} = b = \frac{1-\cos\alpha}{\sin\alpha} = \frac{1-\sqrt{1-\sin^2\alpha}}{\sin\alpha}.$$

Hence, after transformations we have:

$$b\sin\alpha = 1 - \sqrt{1-\sin^2\alpha} \quad \Longrightarrow \quad (1+b^2)\sin\alpha - 2b = 0.$$

Therefore, the solution of the problem in question will be

$$\alpha = \arcsin\frac{2b}{1+b^2}, \tag{30}$$

where b is the ratio between the absciss and the ordinate at the tsunami source point. The circle radius, whose arch is the wave ray trajectory, is now defined from equalities (29):

$$r = y_0\frac{1+b^2}{2b}. \tag{31}$$

Finally, it is possible to write down the equation of the ray that passes the point (x_0, y_0) and the coordinate origin point

$$(x-r)^2 + y^2 = r^2, \qquad 0 < x < x_0. \tag{32}$$

3. Determination of a wave-front line above the bottom slope

We now turn our attention to the wave front determination. In the first paragraph, the solution to the eikonal equation for some model bottom profiles was obtained when a source was situated at the coastal line (the point $(0, 0)$). Let us consider the situation when a point source is located not in the origin of coordinates, but at the point $(0, y_0)$. As an example, let us consider tsunami wave propagation above a uniform bottom slope [3]. Let Ox axis be directed along the rectilinear coastline and Oy axis along seaward. A depth value at any point of the area is determined by the formula $h(x, y) = ky$. It is known that the tsunami wave velocity depends on depth by the Lagrange formula $a = \sqrt{gH}$ (g is acceleration of the gravity, H is water depth) [3]. Due to this, the wave front is described by equation (3) with $\alpha = -1/2$. In Section 2, it was proved that above the bottom slope, wave rays look like segments of a cycloid. This fact makes it possible to derive the equation of a wave front generated by a point source located at a distance y_0 off the shore.

Let us regard a set of cycloids passing through the origin of coordinates depending on radius of the producing circle r. In the parametric form, their equations can be written down as [4]

$$x = ru - r \sin u, \qquad y = r - r \cos u, \tag{33}$$

where u is a parameter. Figure 9 shows one cycloid segment from this set.

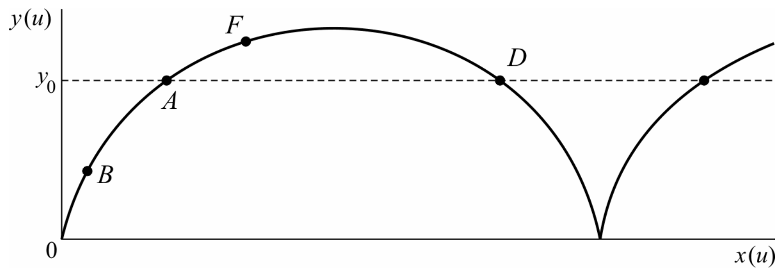

Figure 9

Let $A(x_0, y_0)$ be a point of this segment and u_0 is an appropriate parameter value. We have

$$y_0 = r - r \cos u_0,$$

hence, $\cos u_0 = 1 - \frac{y_0}{r}$ (provided $y_0 \leq 2r$),

$$u_0 = \arccos\Bigl(1 - \frac{y_0}{r}\Bigr), \quad \sin u_0 = \sqrt{1 - \Bigl(1 - \frac{y_0}{r}\Bigr)^2} = \frac{1}{r}\sqrt{y_0(2r - y_0)}$$

(we take a positive value, because the point A is on the left half of the cycloid arc, where $u_0 \le \pi$).

Using (33) we have

$$x_0 = r \arccos\Big(1 - \frac{y_0}{r}\Big) - \sqrt{y_0(2r - y_0)}.$$

Let $t(M, N)$ be a travel time between two arbitrary points M and N of a cycloid. Taking into account

$$ds = \sqrt{(r - r\cos u)^2 + (-r \sin u)^2} = r\sqrt{2(1 - \cos u)},$$
$$v = \sqrt{y} = \sqrt{r(1 - \cos u)},$$

we have

$$t(M, N) = \int_a^b \frac{ds}{v} = \sqrt{2r}b - \sqrt{2r}a,$$

where a, b are values of a parameter as related to the points M, N.

Let us now turn to Figure 9. Let the travel times from the point A to the points $B(x_1, y_1)$, $F(x_2, y_2)$ be equal to C

$$t(B, A) = t(A, F) = C.$$

This means that

$$\sqrt{2r}u_0 - \sqrt{2r}u_1 = C, \qquad \sqrt{2r}u_2 - \sqrt{2r}u_0 = C. \tag{34}$$

From these equalities, with allowance for the expression for u_0, we can find

$$u_1 = \arccos\left(1 - \frac{y_0}{r}\right) - \frac{C}{\sqrt{2r}}, \qquad u_2 = \frac{C}{\sqrt{2r}} + \arccos\left(1 - \frac{y_0}{r}\right). \tag{35}$$

Substituting these values of the parameter into (33), we can find coordinates of the points B and F. If we take another segment of this cycloid (when $\pi < u < 2\pi$), these formulas give us points symmetric to B and F with respect to $y = y_0$ vertical line. Then we regard the cycloid having another radius that also passes the point A. Using similar formulas to (34), (35), we can obtain another pair of points which are located at the same front-line as the first two points B and F. Varying a radius r from $y_0/2$ up to ∞ points obtained by this way, will yield a curve that will be a front line at the time C. Coordinates of points composing a wave isochrone (the front line) are determined by the following formulas:

$$x = \pm\Bigg(\sqrt{y_0(2r - y_0)} - \frac{C\sqrt{r}}{\sqrt{2}} - \sin\Bigg(\arccos\left(1 - \frac{y_0}{r}\right) - \frac{C}{\sqrt{2r}}\Bigg)\Bigg),$$
$$y = r\Bigg(1 - \cos\Bigg(\arccos\left(1 - \frac{y_0}{r}\right) - \frac{C}{\sqrt{2r}}\Bigg)\Bigg).$$

and

$$x = \pm r\left(\sqrt{y_0(2r - y_0)} + \frac{C\sqrt{r}}{\sqrt{2}} - \sin\left(\arccos\left(1 - \frac{y_0}{r}\right) + \frac{C}{\sqrt{2r}}\right)\right),$$
$$y = r\left(1 - \cos\left(\arccos\left(1 - \frac{y_0}{r}\right) + \frac{C}{\sqrt{2r}}\right)\right).$$

Figures 10–12 present results of calculations. Values of parameters are given in figure captions. The position of the point source being marked with an isolated dot. A break in the upper part of the curve is a result of limitation of parameter r. When increasing the upper limit for r, this line break almost disappears.

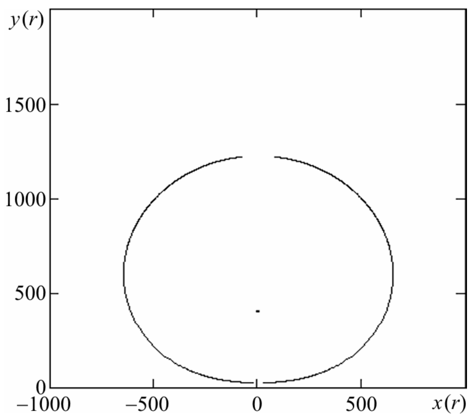

Figure 10. $y_0 = 400$, $C = 30$, $200 \leq r \leq 50000$, $\Delta r = 3$

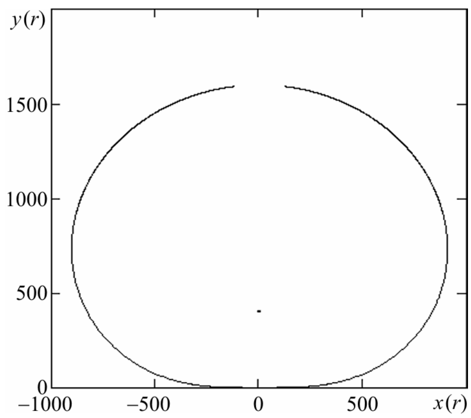

Figure 11. $y_0 = 400$, $C = 45$, $200 \leq r \leq 50000$, $\Delta r = 3$

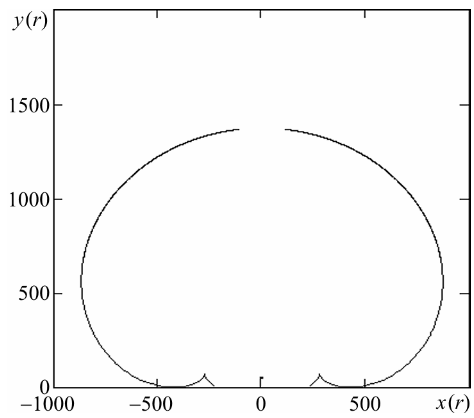

Figure 12. $y_0 = 50$, $C = 55$, $25 \leq r \leq 50000$, $\Delta r = 3$

An unusual behavior of the curve in the last figure (when the source is situated near the shore) can be explained by the following: for small values of r and large values of C, the point F (or B) passes from one arc of the cycloid to another. This case describes a wave reflected from the coast. Thus, in this case, we have a front line of the direct and the reflected waves (a segment between points on the coastline ($y = 0$)).

Conclusion

The solutions obtained can be used for testing numerical methods that are applied to the tsunami wave computations.

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