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Application of schemes based on the differential approximation method to invariant solutions in the gas sphere expansion into vacuum*

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Abstract. This paper considers the problem of obtaining invariant solutions under rotation transformation in the expansion of a gas sphere into vacuum in the Cartesian coordinates for the two-dimensional case. Numerical results are presented and a comparative analysis of conventional and invariant difference schemes is made. Various non-invariant difference schemes of first and second approximation orders are considered. These results show that only the use of an invariant scheme constructed on the basis of the method of differential approximation allows obtaining a correct solution on uniform not dynamic coarse grids.

This paper presents an example of the effectiveness of the differential approximation method [1]. The experienced gained shows that non-invariance as related to a group of transformations, permitted by the original system of differential equations of the difference schemes leads to undesirable calculating effects, which significantly distort the picture of the physical phenomenon in question. Y.I. Shokin [2] obtained a necessary and sufficient condition of invariance of difference schemes in terms of first differential approximation. In [1], specific classes of invariant difference schemes for the two-dimensional equations of gas dynamics are considered, including the proposed number of schemes that are invariant under rotation. The comparative analysis of the results of calculations of the problem of convergence of a spherical shock wave to the center offers an undeniable advantage of using an invariant difference scheme over widespread difference schemes. The proposed method of differential approximation is commonly used for building, analyzing properties and classification difference schemes [3–6].

Along with engineering tasks, there is a wide scope of application of difference schemes for the modelling fundamental problems of astrophysics. Modern mathematical models of astrophysical processes include a description of unsteady and three-dimensional dynamics of a gravitating gas. Despite the existing theory, a such experience of successful application of dif-

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ference methods for solving the gas dynamics equations and for existing ready-made software packages created on their basis, the problem of gravitational gas dynamics requires a special approach. Using the methods of solving hyperbolic systems in different tasks always requires certain criteria to the selection of a method and its modification [7]. The gas dynamics equations are a mathematical expression for fundamental conservation laws of continuous media: mass, momentum and total energy. In the applications of hydrodynamic problems, there often occurs a need to consider additional physical factors such as heat transfer, combustion, gas ionization, the presence of electromagnetic fields, etc. This leads to the necessity of introducing into the equations of new terms and inclusion into the system of additional equations. As a result, the meaning of mathematical models changes: their solution should be interpreted in the new physics terms. This situation takes place for a large class of mathematical models in the modern theoretical astrophysics. Some specific features are typical of gravitational equations of the gas dynamics [8]. In the study of complex astrophysical phenomena, the transition to the modeling of spatial flows of gas is accompanied by manifestation of new physics effects, which are either absent in the problems of a smaller dimension or are presented in minor quantities. In the course of development methods of modeling galactic gas dynamics, most of the existing approaches to the numerical implementation of the gas dynamics equations were tested. Multi-dimensional models lay down special requirements for the numerical methods, which are used to implement them. No less significant factor is a possibility of a fairly simple parallel implementation of the method for calculations on supercomputers.

Currently, from a wide range of numerical methods the following are used: the Lagrangian method SPH (Smoothed Particle Hydrodynamics) and the Euler methods on adaptive grids AMR (Adaptive Mesh Refinement) [9]. The preference exactly of these two approaches is due, primarily, to the desire of researchers to overcome the emerging noninvariance of the solution under the rotation transformation. The Lagrangian SPH is meshless method. This determines a number of its undoubted merits in the case of its application to astrophysical problems. The main disadvantage is a strong dependence of solution on empirical parameters of the method, which results in the need in more laborious but more reliable Eulerian methods. The fact is the problem of a non-invariance solution under the rotation transformation requires for the Euler approach the use of adaptive dynamic grids. More than 20 years ago, adaptive meshes were introduced into practice of modeling. Since that time the technique of numerical construction has become an important part of the astrophysical modeling. In the case of modeling by the Euler grid methods, two basic techniques of adaptive meshes are used: the CR cell refinement and the SAMR block-structured adaptive mesh refinement, which is sometimes called the PR patch refinement [10].

As for the Euler grid method, the way of solving a task is complicated and, also, there arises a problem associated with determination of the quality of meshes [11, 12].

Increasing options of multiprocessor computer systems do not eliminate the problem of search for efficient numerical algorithms. Moreover, the development of computer technology, along with the extension of opportunities for the accuracy and completeness of the formulation of tasks put forward an increasing number of demands for the numerical methods to be used. The proposed in [1] scheme, which is invariant under the rotation transformation, can be a panacea for the problems of computational astrophysics. We will carry out a comparative analysis of the results of calculations of the problem of expansion of a gas sphere into vacuum in the Cartesian coordinates for the two-dimensional case.

1. The problem of expansion of a gas sphere into vacuum

Let us consider a two-dimensional system of gas dynamics in the Cartesian coordinates:

$$\frac{\partial w}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0, \tag{1}$$

where

$$w = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{bmatrix}, \quad f = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u E + u p \end{bmatrix}, \quad g = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v E + v p \end{bmatrix}.$$

Here ρ is gas density, p is pressure, ε and E are internal and total energy, and $\vec{u} = \{u, v\}$ is velocity.

We consider several well-known difference schemes for the 2D gas dynamics equations. For simplicity, we assume a uniform grid $h = h_x = h_y$. Simple stable scheme of first order of accuracy is a upwind scheme. Here is a variant of this scheme:

$$\begin{aligned} \frac{w_{ik}^{n+1} - w_{ik}^n}{\tau} + u_{ik}^n \Delta_x w_{ik}^n + v_{ik}^n \Delta_y w_{ik}^n + \\ w_{ik}^n \left(\frac{u_{i+1,k}^n - u_{i-1,k}^n}{2h} + \frac{v_{i,k+1}^n - v_{i,k-1}^n}{2h} \right) + \\ \frac{F_{i+1,k}^n - F_{i-1,k}^n}{2h} + \frac{G_{i,k+1}^n - G_{i,k-1}^n}{2h} = 0, \end{aligned} \tag{2}$$

where $F_{ik}^n = f_{ik}^n - u_{ik}^n w_{ik}^n$, $G_{ik}^n = g_{ik}^n - v_{ik}^n w_{ik}^n$,

$$\Delta_x w_{ik}^n = \begin{cases} \dfrac{w_{ik}^n - w_{i-1,k}^n}{h}, & u_{ik}^n > 0, \\ \dfrac{w_{i+1,k}^n - w_{ik}^n}{h}, & u_{ik}^n \leq 0, \end{cases} \qquad \Delta_y w_{ik}^n = \begin{cases} \dfrac{w_{ik}^n - w_{i,k-1}^n}{h}, & v_{ik}^n > 0, \\ \dfrac{w_{i,k+1}^n - w_{ik}^n}{h}, & v_{ik}^n \leq 0. \end{cases}$$

As a more complicated scheme of first order, we take the Belotserkovsky–Davydov scheme of large particles [13] and its more invariant modification [14]. We consider only the Lagrangian scheme step, on which the effects of migration, with allowance for the exchange between the cells when they are rebuilding on the old grid Euler, are calculated.

Let us assume that mass flows (larger particles) through the cell boundaries ΔM^n are carried only by the normal to the boundary component of the velocity:

$$w_{ik}^{n+1} h^2 = w_{ik}^n h^2 - \Delta M_{i+1/2,k}^n + \Delta M_{i-1/2,k}^n - \Delta M_{i,k+1/2}^n + \Delta M_{i,k-1/2}^n, \quad (3)$$

where the mass flow is determined by the first order accuracy formulas:

$$\Delta M_{i+1/2,k}^n = \begin{cases} w_{ik}^n \dfrac{u_{ik}^n + u_{i+1,k}^n}{2} h\tau, & u_{ik}^n + u_{i+1,k}^n > 0, \\ w_{i+1,k}^n \dfrac{u_{ik}^n + u_{i+1,k}^n}{2} h\tau, & u_{ik}^n + u_{i+1,k}^n \leq 0. \end{cases}$$

The reliable and well-tested Lucks–Wendroff method of second order accuracy gives excellent results for smooth flows [15]:

$$\begin{gathered} \frac{w_{ik}^{n+1} - \frac{1}{4}\left(w_{i+1,k}^n + w_{i-1,k}^n + w_{i,k+1}^n + w_{i,k-1}^n\right)}{\tau} + \\ \frac{f_{i+1,k}^n - f_{i-1,k}^n}{2h} + \frac{g_{i,k+1}^n - g_{i,k-1}^n}{2h} = 0, \\ \frac{w_{ik}^{n+2} - w_{ik}^{n+1}}{\tau} + \frac{f_{i+1,k}^n - f_{i-1,k}^n}{h} + \frac{g_{i,k+1}^n - g_{i,k-1}^n}{h} = 0. \end{gathered} \quad (4)$$

Invariant patterns proposed in [1] can be interpreted as splitting difference schemes with viscosity, in which at the first step the system of equations of gas dynamics is approximated with second order, at the second step gas dynamics values are adjusted by the difference scheme that approximates the diffusion system of equations. Let us write down one of variants of this set:

$$\begin{gathered} \frac{w_{ik}^* - w_{ik}^n}{\tau} + \frac{f_{i+1,k}^n - f_{i-1,k}^n}{2h} + \frac{g_{i,k+1}^n - g_{i,k-1}^n}{2h} - \\ \tau\left(u_{ik}^n \frac{w_{i+1,k}^n - w_{i-1,k}^n}{2h} + v_{ik}^n \frac{w_{i,k+1}^n - w_{i,k-1}^n}{2h}\right)^2 = 0, \\ \frac{w_{ik}^{n+1} - w_{ik}^*}{\tau} - \frac{2\mu}{\tau^2}\left(w_{i+1,k}^* + w_{i-1,k}^* + w_{i,k+1}^* + w_{i,k-1}^* - 4w_{ik}^*\right) = 0. \end{gathered} \quad (5)$$

By solving the following test problem we compare the results of the calculation by the schemes (2)–(5). At the initial time in the calculation domain we define a circle, which is given by the zero density outside and the unit density inside.

Given the initial density, we can determine all the parameters of the flow from the gas dynamic equations. The initial velocity is zero, and owing to pressure the substance moves through calculation domain in the course of time. On the boundary, homogeneous boundary conditions of the second kind are given. Comparison of the results calculated by difference schemes (2)–(5) with the exact solution of the test problem (Figure 1) shows that all the schemes in question reflect well the qualitative nature of solution.

To analyze the existence of such properties of the solution, as invariance under rotation, we can use contour plots of the density (see Figure 1). The first order accuracy scheme do not allow obtaining axially symmetric solution: both upwind scheme (Figure 1d) and the original method of large particles (Figure 1b). Using schemes of second order accuracy for the transport of gas dynamics values does not make an essential improvement. Despite the fact that the result of calculation by the method of large particles with modifications of defining the schemes velocities at the Lagrangian step [14] is significantly more axially symmetric and with the refinement grid improves, it is not a perfect (Figure 1c). The Lucks–Wendroff scheme of second order of accuracy (4) allows us to obtain significantly better results (Figure 1e)

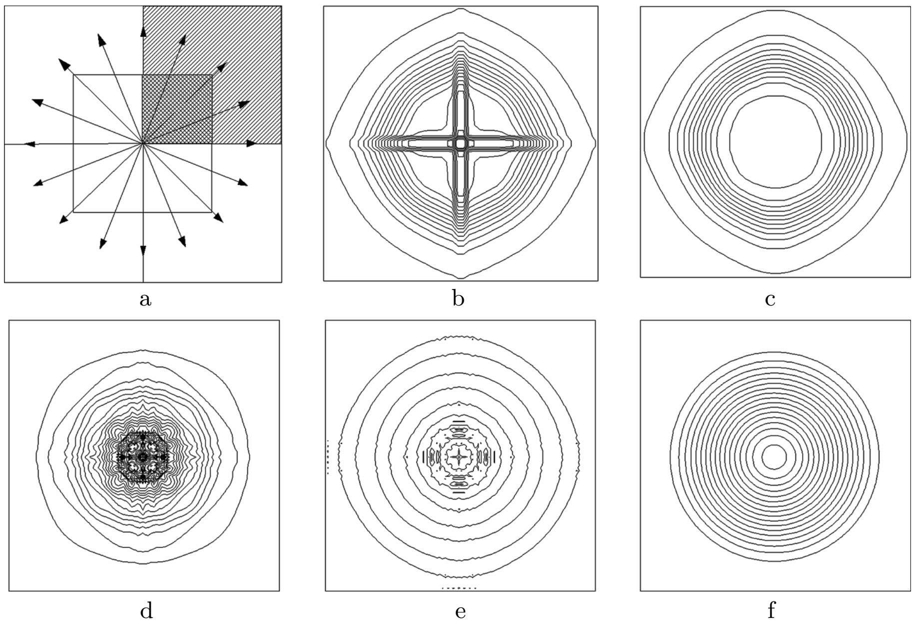
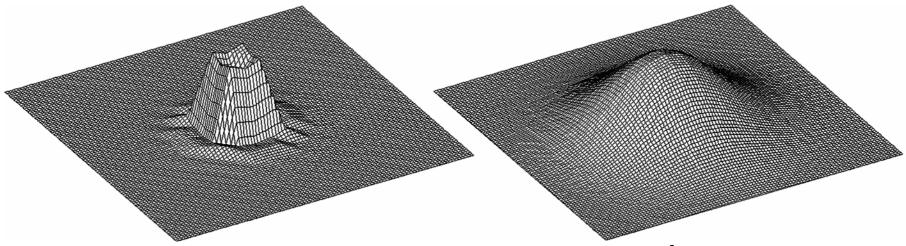


Figure 1. The initial velocity field (a) and contour plots of the density (b)–(f)



a b

Figure 2. Density plots. The results of calculation by the Lucks–Wendroff (a) and the Shokin–Yanenko (b) schemes

which are close to the axially symmetric solution (Figure 1f) obtained by the invariant method (5).

There arises a question: why schemes of high orders of accuracy are not used in solutions of astrophysical problems. The calculations with schemes (2)–(4) on the coarse 16×16 grid do not give us an axially symmetric solution. The density, obtained by the Lucks–Wendroff scheme (4), has vivid features near to the center of the circle (Figure 2a). Obviously, the method reflects the qualitative character of solution. Only invariant scheme (5) allows us to obtain an axially symmetric solution even on the coarse grids (Figure 2b). Scheme (5) is stable for $0 < \mu < 0.25$ and the Courant condition.

The practice of the numerical calculations shows that the method proposed does not require smoothing of the initial data and works well for large density gradients.

2. Conclusion

On the example of the expansion of a gas sphere into vacuum, the effectiveness of the widespread method of differential approximation for constructing, analyzing the properties and classification of difference schemes is shown. Calculations were made with the use of invariant under transformations of rotation scheme obtained with the use of the method of differential approximation. A comparative analysis of the results of calculations of the expansion of a gas sphere into vacuum has shown an indisputable advantage of using an invariant difference scheme as compared to the schemes, both of first and second order accuracy. If a scheme of first order reflects axial symmetry of the solutions only on very fine meshes, the second order scheme expresses the nature of the solution sufficiently well. However, on a coarse grid, the solution is distorted even in the case of a second order scheme. Of course, it is possible to modify such a scheme. In this case, along with a significant complication of the algorithm, the ideal result can not be obtained.

As an example of such a modification, a variant of the algorithm for calculating the movement of gas dynamic values in the scheme of large particles is given. In contrast to the difference schemes, the scheme proposed in [1] reflects the axial symmetry of the solution even on the coarsest grid.

We have thus shown that only the use of an invariant scheme constructed on the basis of a differential approach allows one to obtain a correct solution on the non-dynamic uniform coarse grids. This result is of great importance for solving the problem of finding efficient numerical algorithms for modeling fundamental problems in astrophysics.

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