Bull. Nov. Comp. Center, Num. Anal., 16 (2013), 43–52

Two-level explicit difference schemes*

Yu.M. Laevsky

Abstract. The main disadvantage of explicit schemes for the numerical solution to nonstationary problems is in a very strong stability condition for the size of a time step size. One of the possibilities to improve the efficiency of explicit algorithms is to use different time steps in different space subdomains. From this point of view the methods studied below can be considered as a special case of domain decomposition methods. This approach allows increasing the accuracy of results for the problems corresponding to multiscale physical processes. A striking example is the problem of the laminar flame propagation. There are two natural subdomains in this problem: the subdomain corresponding to the area of diffusion processes and the subdomain corresponding to the kinetic area. The latter is quite narrow and requires a very small spatial step to attain an admissible accuracy. The schemes with the time steps variable in space are studied in [1–3] for the implicit schemes. In [4, 5], a similar technique was applied to provide the localization of a stability condition in subdomains for the explicit schemes. The Dirichlet and the Neumann boundary conditions were used at the interface of the subdomains in [4] and [5], respectively. Applications of these methods are presented in papers [6, 7], and in [8], aspects of parallelization are discussed.

In this paper, we have improved the results from [4]. Namely, we demonstrate the estimate of stability with respect to the right hand-side independent of the number of interior layers (see below).

1. The original family of two-level schemes

Let H_i, $i = 1, 2$, and $H = H_1 \times H_2$ be real finite-dimensional Hilbert spaces with the inner products $(\cdot,\cdot)_i$ and $(\cdot,\cdot) = (\cdot,\cdot)_1 + (\cdot,\cdot)_2$, and with the norms $\|\cdot\|_i$ and $\|\cdot\|$. Let $A_{ii} : H_i \to H_i$, $i = 1, 2$ and $A_{12} : H_2 \to H_1$ be linear continuous operators, and A_{ii} be self-adjoint positive definite in H_i. This means that the inverse operators A_{ii}^{-1} exist. And finally, $A_{12}^T : H_1 \to H_2$ is the adjoint operator to the operator A_{12} with respect to the corresponding inner products:

$$(A_{12}^T u_1, u_2)_2 = (u_1, A_{12} u_2)_1 \quad \forall u_i \in H_i, \ \ i = 1, 2.$$

Let us define the operator $A : H \to H$ as the following matrix operator

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix},$$

*Supported by RFBR under Grant 13-01-00019.

which is the self-adjoint operator in H. Let us assume that A is a positive semi-definite operator. It is well known that this property is provided by the positive semi-definiteness of the Schur complement $S_{22}(A) = A_{22} - A_{12}^T A_{11}^{-1} A_{12}$:

$$(S_{22}(A)u_2, u_2)_2 \geq 0 \qquad \forall u_2 \in H_2.$$

And finally, let

$$\|A_{11}\|_{(1)} \ll \|A_{22}\|_{(2)}.$$

Here and after we denote the norms of operators in H_i as $\|\cdot\|_{(i)}$.

Let us consider a difference scheme in the spaces H_1 and H_2: for given elements $f_1^n \in H_1$, $f_2^{n+\frac{k}{p}} \in H_2$, $n = 0, 1, \ldots$, $k = 0, \ldots, p-1$, $p \geq 2$, $u_i^0 \in H_i$, $i = 1, 2$, let us find the elements $u_1^n \in H_1$, $n = 1, 2, \ldots$, and $u_2^{n+\frac{k}{p}} \in H_2$, $n = 0, 1, \ldots$, $k = 1, \ldots, p$, such that

$$\frac{u_1^{n+1} - u_1^n}{\Delta t} + A_{11}u_1^n + A_{12}u_2^n = f_1^n, \tag{1.1}$$

$$\frac{u_2^{n+\frac{k}{p}} - u_2^{n+\frac{k-1}{p}}}{\tau} + A_{12}^T u_1^{n+\frac{k-1}{p}} + A_{22}u_2^{n+\frac{k-1}{p}} = f_2^{n+\frac{k-1}{p}}, \quad k = 1, \ldots, p, \tag{1.2}$$

$$u_1^{n+\frac{k-1}{p}} = u_1^n + \frac{k-1}{p}\left(u_1^{n+1} - u_1^n\right), \quad k = 1, \ldots, p, \tag{1.3}$$

where $n = 0, 1, 2, \ldots$, $\Delta t = p\tau$.

Let us represent scheme (1.1)–(1.3) in the canonical form in the space H [9] (Samarskii's flavor). To this end, let us exclude from (1.2) the elements with fractional indices taking into account (1.3). As a result we obtain

$$\begin{aligned} u_2^{n+1} = Lu_2^n - \Delta t Q A_{12}^T u_1^n - R A_{12}^T (u_1^{n+1} - u_1^n) + \\ \tau\left(P^{p-1} f_2^n + \ldots + P f_2^{n+\frac{p-2}{p}} + f_2^{n+\frac{p-1}{p}}\right), \end{aligned} \tag{1.4}$$

where

$$P = I_2 - \tau A_{22}, \qquad L = P^p, \qquad Q = \frac{1}{p}(P^{p-1} + \ldots + P + I_2),$$

$$R = \frac{\tau}{p}(P^{p-2} + \ldots + (p-2)P + (p-1)I_2).$$

Here and after I_i and I are the identity operators in H_i and H. It is easy to see that P, L, Q, and R are the self-adjoint operators in H_2 commutating with A_{22}.

Lemma 1.1. *The following presentations are valid*:

$$Q = \frac{1}{\Delta t} A_{22}^{-1}(I_2 - L), \qquad R = A_{22}^{-1}(I_2 - Q).$$

Proof. As $\tau I_2 = A_{22}^{-1}(I_2 - P)$ and $p = \Delta t/\tau$, we have

$$Q = \frac{1}{\Delta t} A_{22}^{-1}(I_2 - P)(P^{p-1} + \ldots + P + I_2) = \frac{1}{\Delta t} A_{22}^{-1}(I_2 - L).$$

Then

$$\begin{aligned} R &= \frac{1}{p} A_{22}^{-1}(I_2 - P)(P^{p-2} + \ldots + (p-2)P + (p-1)I_2) \\ &= \frac{1}{p} A_{22}^{-1}(pI_2 - P^{p-1} - \ldots - P - I_2) = A_{22}^{-1}(I_2 - Q). \end{aligned} \qquad \square$$

Let the inverse operator Q^{-1} exist. Further we will present sufficient conditions for this. According to Lemma 1.1, $A_{22} = Q^{-1}(I_2 - L)/\Delta t$, and equality (1.4) can be transformed to:

$$Q^{-1} R A_{12}^T \frac{u_1^{n+1} - u_1^n}{\Delta t} + Q^{-1} \frac{u_2^{n+1} - u_2^n}{\Delta t} + A_{12}^T u_1^n + A_{22} u_2^n = \varphi_2^n, \tag{1.5}$$

where

$$\varphi_2^n = \frac{1}{p} Q^{-1} \Big(P^{p-1} f_2^n + \ldots + P f_2^{n+\frac{p-2}{p}} + f_2^{n+\frac{p-1}{p}} \Big). \tag{1.6}$$

According to (1.1), (1.5), we obtain the two-layer scheme in the canonical representation

$$B \frac{u^{n+1} - u^n}{\Delta t} + A u^n = \varphi^n, \tag{1.7}$$

where $\varphi^n = (\varphi_1^n, \varphi_2^n)^T \in H$, $\varphi_1^n = f_1^n$, the element φ_2^n is defined by (1.6), $B : H \to H$ is the following matrix operator

$$B = \begin{pmatrix} I_1 & O_{12} \\ Q^{-1} R A_{12}^T & Q^{-1} \end{pmatrix}. \tag{1.8}$$

Here and after $O_{ij} : H_j \to H_i$ is a null operator which converts any element from the space H_j into a null element from the space H_i.

Our objective is to study the stability of the scheme (1.7). This issue has a close connection with properties of certain polynomials.

2. Auxiliary polynomials

On the closed interval $[0, 1]$, let us consider the functions

$$l(x) = (1 - x)^p, \qquad q(x) = \frac{1}{x}\Big(1 - l(x)\Big). \tag{2.1}$$

Lemma 2.1. *The function $q(x)$ being the polynomial of degree $p-1$, $q(0)=p$, $q(1)=1$, and the following inequalities hold at $x \in [0,1]$:*

$$1 \le q(x) \le p, \qquad q(x) \ge p\, l(x), \qquad q(x) \ge k(1-x)^{k-1}, \quad k = 2, \dots, p.$$

Proof. Let $y = 1-x$. According to (2.1)

$$q(x) = \frac{1-y^p}{1-y} = 1 + y + \ldots + y^{p-1}.$$

From this equality, it follows that $q(x)$ is the polynomial of degree $p-1$, $q(0) = p$ $(y=1)$, $q(1) = 1$ $(y=0)$. It is evident that $1 \le q(x) \le p$ at $y \in [0,1]$, and

$$q(x) - p\, l(x) = 1 + y + \ldots + y^{p-1} - p\, y^p = \sum_{k=0}^{p-1} y^k (1 - y^{p-k}) \ge 0,$$

that is, the first and the second inequalities from the Lemma statement are valid. Then

$$\begin{aligned} q(x) - k(1-x)^{k-1} &\ge 1 + y + \ldots + y^{k-1} - k y^{k-1} \\ &= \sum_{l=1}^{k} y^{l-1}(1 - y^{k-l}) \ge 0, \quad k = 2, \dots, p, \end{aligned}$$

and the latter is proved. □

In addition to $l(x)$ and $q(x)$, we use the function

$$r(x) = \frac{1}{x}(p - q(x)). \tag{2.2}$$

Let $y = 1-x$. Then

$$r(x) = \frac{1}{1-y}(p - 1 - y - \ldots - y^{p-1}) = p - 1 + (p-2)y + \ldots + y^{p-2},$$

that is, $r(x)$ is the polynomial of degree $p-2$.

Let ρ be a positive number. In the interval $[0, \rho]$, let us consider the polynomials

$$\hat{l}(\lambda) = l(\lambda/\rho), \qquad \hat{q}(\lambda) = \frac{1}{p}\, q(\lambda/\rho), \qquad \hat{r}(\lambda) = \frac{1}{p\rho}\, r(\lambda/\rho). \tag{2.3}$$

Based on these polynomials, let us define the operators $\hat{l}(A_{22})$, $\hat{q}(A_{22})$ and $\hat{r}(A_{22})$ acting in the space H_2.

Lemma 2.2. *At $\rho\tau = 1$, the following equalities are valid:*

$$L = \hat{l}(A_{22}), \quad Q = \hat{q}(A_{22}), \quad R = \hat{r}(A_{22}),$$

where the operators L, Q, and R are defined above.

Proof. According to (2.1), (2.3)

$$\hat{l}(A_{22}) = l(\tau A_{22}) = (I_2 - \tau A_{22})^p = L.$$

From Lemma 1.1 it follows that

$$\hat{q}(A_{22}) = \frac{1}{p}\, q(\tau A_{22}) = \frac{1}{p}\,(\tau A_{22})^{-1}(I_2 - l(\tau A_{22})) = \frac{1}{\Delta t}\, A_{22}^{-1}(I_2 - L) = Q,$$

and according to (2.2)

$$\hat{r}(A_{22}) = \frac{\tau}{p}\,(\tau A_{22})^{-1}(pI_2 - q(\tau A_{22})) = A_{22}^{-1}(I_2 - Q) = R. \qquad \square$$

Corollary. *Let the condition*

$$\tau \|A_{22}\|_{(2)} \le 1 \tag{2.4}$$

hold. Then Q is a positive definite operator.

Proof. As A_{22} is a self-adjoint operator, its eigenvalues are real, and eigenvectors compose a full orthogonal system in H_2. If λ is an eigenvalue of the operator A_{22}, then $\hat{q}(\lambda)$ is the eigenvalue of the operator Q according to Lemma 2.2. As $0 < \lambda \le \|A_{22}\|_{(2)}$ (A_{22} is positive definite), it follows that $\tau\lambda \in [0, 1]$ from condition (2.3), and, for the values $x = \tau\lambda$, Lemma 2.1 is valid. Then $q(\tau\lambda) \ge 1$, and according to (2.3) $\hat{q}(\lambda) \ge 1/p$, that is, the eigenvalues of the operator Q are positive. This means that Q is a positive definite operator. $\square$

From this statement follow the existence of the operator Q^{-1} and representation (1.8) of the matrix operator B.

3. Stability with respect to the initial data

Let us study the stability of scheme (1.7) with respect to the initial data assuming φ^n is the null-element of the space H.

Theorem 3.1. *Let condition* (2.4) *be valid, and*

$$\Delta t \|A_{11}\|_{(1)} \le 1. \tag{3.1}$$

Then for any $u \in H$ the inequality

$$(Bu, u) \ge \frac{\Delta t}{2}\Big(Au, u\Big) \tag{3.2}$$

holds, and the estimate

$$(Au^n, u^n) \le (Au^0, u^0), \quad n = 1, 2, \ldots \tag{3.3}$$

is valid.

Proof. As the space H is real, $(Bu, u) = (B_0 u, u)$ for any $u \in H$, where $B_0 = \frac{1}{2}(B + B^T)$, B^T is the adjoint operator to B. Therefore, to prove inequality (3.2) it is sufficient to prove the positive definiteness of the operator $D = B_0 - \frac{1}{2}\Delta t A$. It is not difficult to show that D can be represented in the form

$$D = D' + D'',$$

where

$$D' = \begin{pmatrix} I_1 - \Delta t A_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix}, \qquad D'' = \frac{\Delta t}{2} \begin{pmatrix} A_{11} & A_{12} G \\ G A_{12}^T & \frac{2}{\Delta t} Q^{-1} - A_{22} \end{pmatrix},$$

$G = \frac{1}{\Delta t} Q^{-1} R - I_2$. It is easy to see that condition (3.1) provides positive semi-definiteness of the operator D'. As A_{11} is positive definite, to prove the positive semi-definiteness of the operator D'' we can prove positive semi-definiteness of the Schur complement

$$S_{22}(D'') = Q^{-1} - \frac{\Delta t}{2} A_{22} - \frac{\Delta t}{2} G A_{12}^T A_{11}^{-1} A_{12} G = \widetilde{S}_{22} + \frac{\Delta t}{2} G S_{22}(A) G,$$

where

$$\widetilde{S}_{22} = Q^{-1} - \frac{\Delta t}{2} A_{22} - \frac{\Delta t}{2} A_{22} G^2.$$

As is mentioned above, the Schur complement $S_{22}(A) = A_{22} - A_{12}^T A_{11}^{-1} A_{12}$ is positive semi-definite. This means that to prove the positive semi-definiteness of the operator $S_{22}(D'')$, we need to prove the positive semi-definiteness of the operator $\widetilde{S}_{22}$. To this end we need to prove that the function is non-negative

$$\psi(\lambda) = \hat{q}^{-1}(\lambda) - \frac{\Delta t \lambda}{2} - \frac{\Delta t \lambda}{2} \left(\frac{1}{\Delta t} \hat{q}^{-1}(\lambda) \hat{r}(\lambda) - 1 \right)^2, \quad \lambda \in [0, \lambda_{\max}],$$

where $\lambda_{\max} = \|A_{22}\|_{(2)}$. Using formulas (2.3) at $\rho\tau = 1$ and $x = \tau\lambda$, we obtain the equality

$$\psi(\lambda) = \frac{px}{2(1 - l(x))^2} \left[1 - l^2(x) - (q(x)/p - l(x))^2 \right].$$

According to condition (2.3) $x \in [0, 1]$, and we may use Lemma 2.1. Then

$$l^2(x) + (q(x)/p - l(x))^2 \le \frac{1}{p^2} q^2(x) \le 1.$$

Therefore, the function $\psi(\lambda)$ is non-negative for $\lambda \in [0, \lambda_{\max}]$. This leads to the positive semi-definiteness of the operator $\widetilde{S}_{22}$. Thus D'' is a positive semi-definite operator, and according to the above arguments, inequality (3.2) is proved. And finally, the stability inequality is a direct consequence of (3.2) and [9]. $\square$

Remark. Condition (2.4) can be rewritten in the following form:

$$p \geq \Delta t \|A_{22}\|_{(2)}.$$

Therefore, the stability conditions with respect to the initial data for the two-level explicit scheme are condition (3.1) for the step Δt, which depends only on the norm of the operator A_{11}, and the condition for the number p of interior steps, which depends only on the norm of the operator A_{22}.

4. Stability with respect to the right-hand side

As is well known, from the stability with respect to initial data, the stability with respect to the right-hand side follows (but with definite norms harmonization) [9]. Namely, the inequality of stability contains the value $(AB^{-1}\varphi^n, B^{-1}\varphi^n)$. The operator B does not commute with the operator A. This difficulty can be overcome with the usage of a stronger stability condition instead of (3.1). It is known [9] that for schemes like (1.7) with vanishing initial data and with the inequality

$$(Bu, u) \geq \frac{1}{2}(\varepsilon I + \Delta t A) \quad \forall u \in H, \quad \varepsilon > 0, \tag{4.1}$$

being valid, the following estimate holds:

$$(Au^m, u^m) \leq \frac{1}{\varepsilon} \sum_{n=0}^{m-1} \Delta t \|\varphi^n\|^2. \tag{4.2}$$

Our aim is to obtain a condition for inequality (4.1) and to estimate the norm $\|\varphi^n\|$ with the norms $\|f_1^n\|_1$ and $\|f_2^{n+\frac{k}{p}}\|_2$.

Theorem 4.1. *Let condition* (2.4) *be valid, and*

$$\Delta t \|A_{11}\|_{(1)} \leq 1 - \varepsilon, \;\; \varepsilon \in (0, 1). \tag{4.3}$$

Then for the vanishing initial data, the inequality

$$(Au^m, u^m) \leq \frac{1}{\varepsilon} \sum_{n=0}^{m-1} \Delta t \Big(\|f_1^n\|_1^2 + 4 \max_{k=1,\dots,p} \|f_2^{n+\frac{k-1}{p}}\|_2^2 \Big) \tag{4.4}$$

holds.

Proof. Let D_0 be a self-adjoint positive definite operator in H. Using $\varepsilon/\Delta t$-inequality, we obtain the following:

$$2|(\varphi^n, v)| = 2|(D_0^{-1/2}\varphi^n, D_0^{1/2}v)| \le \frac{\Delta t}{\varepsilon}(D_0^{-1}\varphi^n, \varphi^n) + \frac{\varepsilon}{\Delta t}(D_0 v, v).$$

Then multiplying equality (1.7) by $2v \equiv 2(u^{n+1} - u^n)$ and using the latter inequality, we obtain the estimate

$$(Au^m, u^m) \le \frac{1}{\varepsilon}\sum_{n=0}^{m-1} \Delta t\ (D_0^{-1}\varphi^n, \varphi^n) \tag{4.5}$$

provided that

$$(Dv, v) \equiv (B_0 v, v) - \frac{\varepsilon}{2}(D_0 v, v) - \frac{\Delta t}{2}(Av, v) \ge 0 \quad \forall v \in H, \tag{4.6}$$

where $B_0 = \frac{1}{2}(B + B^T)$. Let us formulate inequality (4.6) assuming

$$D_0 = \begin{pmatrix} I_1 & O_{12} \\ O_{21} & Q^{-1} \end{pmatrix}.$$

Similar to the proof of Theorem 3.1, we use the representation

$$D = D' + D'',$$

where

$$D' = \left(1 - \frac{\varepsilon}{2}\right)\begin{pmatrix} I_1 - \frac{1}{1-\varepsilon}\Delta t A_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix},$$

$$D'' = \frac{\Delta t}{2}\begin{pmatrix} \frac{1}{1-\varepsilon}A_{11} & A_{12}G \\ GA_{12}^T & \frac{2-\varepsilon}{\Delta t}Q^{-1} - A_{22} \end{pmatrix},$$

$G = \frac{1}{\Delta t}Q^{-1}R - I_2$. According to (4.3), D' is a positive semi-definite operator. Similar to the proof of Theorem 3.1, the sufficient condition for the positive semi-definiteness of the operator D'' is the positive semi-definiteness of the operator

$$\widetilde{S}_{22} = (2-\varepsilon)Q^{-1} - \Delta t\, A_{22} - (1-\varepsilon)\Delta t\, A_{22}G^2.$$

The latter follows from the non-negative function

$$\psi(x) = \frac{2-\varepsilon}{1-l(x)} - 1 - \frac{1-\varepsilon}{(1-l(x))^2}\left(q(x)/p - l(x)\right)^2, \quad x \in (0,1).$$

As

$$l^2(x) + \left(q(x)/p - l(x)\right)^2 \le 1$$

(see the proof of Theorem 3.1), then

$$\psi(x) \geq \frac{2-\varepsilon}{1-l(x)} - 1 - (1-\varepsilon)\frac{1+l(x)}{1-l(x)} = \frac{\varepsilon l(x)}{1-l(x)} \geq 0.$$

Therefore, inequality (4.6) is proved. Now we estimate the inner product

$$(D_0^{-1}\varphi^n, \varphi^n) = \|f_1^n\|_1^2 + \|Q^{1/2}\varphi_2^n\|^2. \tag{4.7}$$

According to formula (1.6), we have

$$\|Q^{1/2}\varphi_2^n\| \leq \frac{1}{p} \max_{k=1,\dots,p} \|f_2^{n+\frac{k-1}{p}}\|_2 \sum_{k=1}^{p} \|Q^{-1/2}P^{k-1}\|_{(2)} .$$

Using the third inequality from Lemma 2.1, we obtain

$$\|Q^{-1/2}P^{k-1}\|_{(2)}^2 = \|Q^{-1}P^{2(k-1)}\|_{(2)} \leq \|Q^{-1}P^{k-1}\|_{(2)} \leq \frac{p}{k}, \quad k = 1, \dots, p.$$

Then

$$\frac{1}{p} \sum_{k=1}^{p} \|Q^{-1/2}P^{k-1}\|_{(2)} \leq \frac{1}{\sqrt{p}} \sum_{k=1}^{p} \frac{1}{\sqrt{k}} \leq 2.$$

The latter inequality leads to the estimate

$$\|Q^{1/2}\varphi_2^n\| \leq 2 \max_{k=1,\dots,p} \|f_2^{n+\frac{k-1}{p}}\|_2.$$

From (4.5), (4.7), follows inequality (4.4). □

References

[1] Ewing R.E., Lazarov R.D., Pasciak J.E., Vassilevski P.P. Finite Element Methods for Parabolic Problems with Time Steps Variable in Space. — Wyoming, 1989. — (Report / Univ. of Wyoming; 1989-05).

[2] Matus P.P. On the difference schemes with composite meshes for nonstationary problems of mathematical physics // Differential Equations. — 1990. — Vol. 26, No. 7. — P. 1241–1254 (In Russian).

[3] Drobyshevich V.I., Laevsky Yu.M. An algorithm of solution of parabolic equations with different time-steps in subdomains // Russ. J. Numer. Anal. and Math. Model. — 1992. — Vol. 7, No. 3. — P. 205–220.

[4] Laevsky Yu.M., Banushkina P.V. The compound explicit schemes // Siberian J. Numer. Mathematics. — 2000. — Vol. 3, No. 2. — P. 165–180 (In Russian).

[5] Banushkina P.V., Laevsky Yu.M. Multilevel explicit schemes and their stability // Russ. J. Numer. Anal. and Math. Model. — 2001. — Vol. 16, No. 3. — P. 215–233.

[6] Zotkevich A.A., Laevsky Yu.M. Numerical simulation of laminar flame propagation based on two-level explicit difference schemes // Computational Technologies. — 2006. — Vol. 11, No. 6. — P. 31–43 (In Russian).

[7] Laevsky Yu.M., Yausheva L.V. Simulation of filtrational gas combustion processes in nonhomogeneous porous media // Numerical Analysis and Applications. — 2009. — Vol. 2, No. 2. — P. 140–153.

[8] Laevsky Yu.M., Banushkina P.V., Litvinenko S.A., Zotkevich A.A. Parallel algorithms for non-stationary problems: survey of new generation of explicit schemes // Lecture Notes in Computer Science. — Springer, 2001. — Iss. 2127. — P. 442–446.

[9] Samarskii A.A., Gulin A.V. Stability of Difference Schemes. — Moscow: Nauka, 1973 (In Russian).