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On numerical experiments with some iterative solvers in mixed finite element method*

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1. Introduction

This paper presents some numerical experiments with iterative solvers of algebraic linear systems for mixed finite element approximations [1]. We consider the following problem: Ω is a parallelepipedal domain,

$$\nabla(k\nabla u) = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_D, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_N,$$

where $\Gamma_D \cup \Gamma_N = \partial\Omega$. The domain consists of three kinds of parallelepipeds: a) inclusive, $k = 1$; b) small internal parallelepipeds, where k has a small value, and, finally, c) the internal parallelepiped-bridges, where k has a big value. Our objective is to adapt Conjugate Gradient method (CG) to a problem with a large number of degrees of freedom and large jumps in coefficients. This means that we design a preconditioner for the CG with very simple implementation on the one hand, and with convergence nearly independent of the number of degrees of freedom and of the coefficients jumps, on the other. The mathematical basis for such a preconditioner is a well-known additive Swartz method [2]. The efficiency of a preconditioner is demonstrated on a number of sample problems that include the large jumps in the coefficients.

The use of the Raviart–Thomas lower order elements [1] gives us the following algebraic system

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} p \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

After elimination of the vector p we arrive to the system $B^T A^{-1} B u = -f$ and design a preconditioner for the matrix of this system. Note that the inversion of the matrix A can be exactly implemented (Ω is a parallelepiped).

2. The additive Swartz preconditioner

Let us construct the preconditioner M^{-1} presented as follows (according to the additive Swartz method): $M^{-1} = R\Lambda R^T + D$.

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This matrix is defined on the original mesh $\{x_i, y_i, z_i\}$ which generates the grid space V. Let us introduce the auxiliary grid $\{X_i, Y_i, z_i\}$ with new nodes in x- and y-directions and with steps $h_x = l_x/2^{mx}$ and $h_y = l_y/2^{my}$, respectively, where l_x and l_y are the corresponding sizes of the inclusive parallelepiped. Moreover, $h_x \geq \max(x_{i+1} - x_i)$ and $h_y \geq \max(y_{i+1} - y_i)$. This means that the auxiliary grid is coarser than the original one. Therefore, instead of the auxiliary grid we use the coarse grid. Such an approach was proposed in [2, 3]. The coarse grid is uniform in x- and y-directions, and we can use the DFT (the Discrete Fourier Transform) for these directions. Let V_0 be a space corresponding to the coarse grid. Then we construct the matrix $R : V_0 \to V$ using the primitive bilinear interpolation, and the main operation is to invert the grid Laplace type 7-point operator with the use of the DFT in x and y-directions and LU-factorization in z-direction. However, because of $\dim V_0 \leq \dim V$, the matrix $R\Lambda R^T$ can be singular, with a nontrivial kernel. Moreover, while constructing the matrix Λ, we take into account only z-layer — the subdomains $[x_0, x_0 + l_x] \times [y_0, y_0 + l_y] \times [z', z'']$. Therefore, we add the diagonal matrix D with the positive entries which corresponds to the first step of the simple Jacobi iterations. This means that we take into account the coefficient jumps in the matrix D.

3. Tests for the model problems

3.1. Description of the model problems. Let Ω be the unit cube with a small parallepipedal subdomain ω and $G = \Omega \setminus \omega$. In G, $k = 1$ and, in ω, $k \neq 1$. An integer n specifies the grid step $h = 1/n$.

In Tables 1–4 the experiments with 4 different subdomains are shown. A number of iterations is given in the form "i (j)" where i and j correspond to the preconditioners without and with taking into account the coefficient jumps in the diagonal matrix D (see Section 2). In Tables 5–6, we demonstrate the influence of a non-uniform and an anisotropic mesh. In Tables 1–6 a small coefficient in ω was considered. But most difficulties happen with a large coefficient. In Tables 7–8, the results for large coefficients are given. For a subdomain, which is a thin plate or a thin plate with a small hole, we used the preconditioner with allowance for the coefficient jumps when factorizing a grid operator on a coarse grid.

Table 1. The number of iterations for $\omega = [0.45, 0.55]^3$

n	$k = 0.1$	$k = 0.01$	$k = 0.001$
20	21 (19)	31 (19)	29 (19)
40	26 (20)	42 (20)	31 (20)
60	27 (20)	42 (20)	31 (19)
80	29 (23)	37 (24)	33 (21)

Table 2. The number of iterations for $\omega = [0.45, 0.55]^2 \times [0, 1]$

n	$k = 0.1$	$k = 0.01$	$k = 0.001$
20	25 (20)	41 (20)	42 (20)
40	29 (22)	46 (22)	48 (22)
60	33 (24)	46 (26)	35 (21)
80	34 (26)	50 (27)	37 (26)

Table 3. The number of iterations for $\omega = [0,1] \times [0.45, 0.55]^2$

n	$k = 0.1$	$k = 0.01$	$k = 0.001$
20	24 (20)	41 (21)	42 (21)
40	30 (22)	54 (23)	48 (23)
60	33 (23)	53 (24)	46 (23)
80	35 (26)	58 (27)	51 (27)

Table 4. The number of iterations for $\omega = [0.25, 0.75]^3$

n	$k = 0.1$	$k = 0.01$	$k = 0.001$
20	39 (27)	78 (31)	72 (29)
40	40 (35)	76 (44)	78 (44)
60	37 (33)	66 (52)	64 (53)
80	39 (37)	71 (61)	70 (57)

Table 5. The number of iterations for $\omega = [0.433, 0.567]^3$, $h_{\max}/h_{\min} = 2$

n	$k = 0.1$	$k = 0.01$	$k = 0.001$
20	52 (35)	101 (35)	79 (35)
40	64 (38)	100 (39)	86 (38)
60	53 (36)	87 (41)	53 (35)
80	64 (41)	102 (48)	53 (40)

Table 6. The number of iterations for $\omega = [0.46, 0.54]^3$, $h_{\max}/h_{\min} = 4$

n	$k = 0.1$	$k = 0.01$	$k = 0.001$
20	94 (78)	158 (78)	174 (78)
40	130 (88)	216 (88)	138 (88)
60	136 (81)	206 (81)	127 (79)
80	164 (97)	257 (97)	152 (95)

Table 7. The number of iterations for $\omega = [0,1]^2 \times [0.45, 0.55]$

n	$k = 10$	$k = 100$	$k = 1000$
20	29 (22)	82 (24)	223 (24)
40	39 (22)	122 (23)	351 (24)
60	41 (20)	133 (21)	296 (22)
80	42 (22)	— (23)	— (23)

Table 8. The number of iterations for $\omega = [0,1]^2 \times [0.45, 0.55] \setminus [0.45, 0.55]^3$

n	$k = 10$	$k = 100$	$k = 1000$
20	29 (22)	87 (24)	241 (24)
40	39 (24)	127 (27)	381 (27)
60	42 (24)	144 (30)	334 (32)
80	41 (27)	— (37)	— (40)

Finally, in Tables 9–13, the condition numbers for the preconditioned matrix are presented (using the coefficient jumps in the matrix D and the layers with large coefficients when inverting a grid operator on a coarse grid). In all the tests we assigned the zero Dirichlet boundary condition on the lower boundary of Ω, the rest of ω boundaries had the zero Neumann condition. The entries of the right-hand-side vector are nullified except the first and the last one that are equal to 1. To solve these problems, the SSG with our preconditioner are used. Stopping criterium — the residual relations — is equal to 10^{-5}.

Table 9. The condition number with non-modified preconditioner for $\omega = [0.45, 0.55]^3$

n	$k = 1$	$k = 0.1$	$k = 0.01$	$k = 0.001$
20	12	25	236	2342
40	12	61	601	5990
60	12	92	913	—

Table 10. The condition number for $\omega = [0.45, 0.55]^3$

n	$k = 0.1$	$k = 0.01$	$k = 0.001$
20	12	12	12
40	17	20	21
60	29	44	47

Table 11. The condition number for $\omega = [0.45, 0.55]^2 \times [0, 1]$

n	$k = 0.1$	$k = 0.01$	$k = 0.001$
20	14	15	15
40	23	30	31
60	37	64	69

Table 12. The condition number for $\omega = [0, 1]^2 \times [0.45, 0.55]$

n	$k = 0.1$	$k = 0.01$	$k = 0.001$
20	17	22	23
40	17	22	23
60	14	19	20

Table 13. The condition number for $\omega = [0, 1]^2 \times [0.45, 0.55] \setminus [0.45, 0.55]^3$

n	$k = 0.1$	$k = 0.01$	$k = 0.001$
20	17	22	23
40	21	31	32
60	32	57	63

3.2. Numerical results. The tables presented show that with a fixed geometry (as described in Section 1), if the following conditions hold, the convergence rate slightly depends on the grid dimensions:

- every parallelepiped-bridge in Ω has huge values of k;
- in every small parallelepiped inside Ω, k is relatively small.

Also, one can see, that if a condition does not hold, the described preconditioner gives poor results (for example, if k in a bridge is small, no good result is expected).

References

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