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# Stability analysis of a continuation problem for the Helmholtz equation*

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**Abstract.** We investigate the continuation problem for the elliptic equation. The continuation problem is formulated in the operator form $Aq = f$. Singular values of the operator $A$ are presented and analyzed for the continuation problem for the Helmholtz equation. Results of numerical experiments are presented.

**Keywords:** Helmholtz equation, inverse problem, singular values, degree of ill-posedness.

## 1. Introduction

The Cauchy problem for the Helmholtz equation is a well-known example of an ill-posed problem. The solution is unique, but does not depend continuously on the Cauchy data in standard norms [1, 6, 7].

The Cauchy problem for the Helmholtz equation was theoretically investigated by Fritz John in [6]. He has shown that the conditional stability estimate for $k$ is the best logarithmic estimate. It was demonstrated in [4,5] that ill-posedness of the Cauchy problem for the Helmholtz equation strictly depends on the wave number $k$ and increases as $k$ increases. It was shown that at some restrictions on the convexity of domains, the conditional stability estimate leads to limitations that depend on the wave number. There is a subspace of the data space in which the Cauchy problem is well-posed, and this subspace grows with larger $k$ (a subspace of stability). For more general geometries, the authors studied the ill-posedness by computing singular values of some operators associated with corresponding well-posed (direct) boundary value problems.

Numerical calculations using various regularization methods are presented, for instance, in the following papers: quasi-reversibility method [2, 10], frequency space cut-off [15], iterative methods [9, 11, 12], regularization methods [3, 13, 14, 16].

It was shown [16] that the Cauchy problem for the Helmholtz equation depends on different smoothness situations, the best possible accuracy may be of the Holder type, of the logarithmic type, or of some other type.

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We consider the continuation problem

$$\begin{aligned} & u_{xx} + L(y)u = 0, && (x,y) \in \Omega, \\ & u(0,y) = f(y), && y \in \mathcal{D}, \\ & u_x(0,y) = 0, && y \in \mathcal{D}, \\ & u|_{\partial\mathcal{D}} = 0, && x \in (0,h), \end{aligned} \tag{1}$$

in the domain $\Omega := \{(x,y) \in \mathbb{R}^{n+1} : \ x \in (0,h), \ y \in \mathcal{D} \subset \mathbb{R}^n\}$ with the matched condition

$$f|_{\partial\mathcal{D}} = 0, \qquad x \in (0,h). \tag{2}$$

Here

$$L(y)u = \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial u}{\partial y_j}\right) - c(y)u,$$

$\mathcal{D}$ is a connected bounded domain with the Lipschitz boundary, and the operator $L(y)$ has the following properties:

$$M_1 \sum_{j=1}^{n} \nu_j^2 \leq \sum_{i,j=1}^{n} a_{ij}(y)\nu_i\nu_j, \quad \forall \nu_i \in \mathbb{R}, \qquad a_{ij} = a_{ji}, \quad i,j = 1,\ldots,n,$$
$$0 \leq c(y) \leq M_2, \quad c \in C(\overline{\mathcal{D}}), \quad a_{ij} \in C^1(\overline{\mathcal{D}}),$$

where $M_1$ and $M_2$ are positive constants. We will denote by $M$ all the constants which do not depend on the main parameters of inequalities.

In Section 2, we describe and justify a gradient type algorithm for the continuation problem (1), (2).

In Section 3, we investigate the continuation problem for the Helmholtz equation for simple geometries and homogeneous media.

In Section 4, the numerical results are presented and analyzed.

## 2. Optimization approach

Let us consider ill-posed continuation problem (1), (2) as inverse problem to the following direct problem

$$\begin{aligned} & u_{xx} + L(y)u = 0, && (x,y) \in \Omega, \\ & u_x(0,y) = 0, && y \in \mathcal{D}, \\ & u(h,y) = q(y), && y \in \mathcal{D}, \\ & u|_{\partial\mathcal{D}} = 0, && x \in (0,h), \end{aligned} \tag{3}$$

with the matched condition

$$q|_{\partial\mathcal{D}} = 0, \qquad x \in (0,h), \tag{4}$$

In the direct problem (3), it is required to determine $u(x, y)$ in $\Omega$ from the function $q(y)$ given on a part of the boundary $x = h$ of the domain $\Omega$.

The inverse problem consists in finding $q(y)$ from (3) and the additional information

$$u(0, y) = f(y). \tag{5}$$

Inverse problem (3) and continuation problem (1), (2) are equivalent. If we solve the inverse problem, we find the solution of the continuation problem $u(x, y)$ and *vice versa*.

**Theorem 1** (the well-posedness of direct problem [7]). *If $q \in L_2(\mathcal{D})$, then the direct problem* (3), (4) *has a unique generalized solution $u \in L_2(\Omega)$ such that*

$$\|u\|_{L_2(\Omega)} \le M\|q\|_{L_2(\mathcal{D})}.$$

**Theorem 2** (existence of a trace). *If $q \in L_2(\mathcal{D})$, then the solution to direct problem* (3), (4) *has a trace $u(0, y) \in L_2(\mathcal{D})$ and the following estimate holds*:

$$\|u(0, y)\|_{L_2(\mathcal{D})} \le M\|q\|_{L_2(\mathcal{D})}.$$

**Theorem 3** (conditional stability estimate [7, 8]). *Let $q, f \in L_2(\mathcal{D})$. If continuation problem* (1), (2) *has a solution $u \in C^2(\overline{\Omega})$, then it satisfies the inequality*

$$\int_{\mathcal{D}} u^2(x, y)\, dy \le \|q\|^{2x/h}_{L_2(\mathcal{D})} \|f\|^{2(h-x)/h}_{L_2(\mathcal{D})}, \quad x \in (0, h).$$

Along with direct problem (3), (4) we consider the adjoint problem

$$\begin{aligned}
&\psi_{xx} + L(y)\psi = 0, && (x, y) \in \Omega,\\
&\psi_x(0, y) = \mu(y), && y \in \mathcal{D},\\
&\psi(h, y) = 0, && y \in \mathcal{D},\\
&\psi|_{\partial\mathcal{D}} = 0, && x \in (0, h).
\end{aligned} \tag{6}$$

Here it is required to determine $\psi(x, y)$ from the given function $\mu(y)$.

**Theorem 4** (the well-posedness of the adjoint problem [7]). *If $\mu \in L_2(\mathcal{D})$, then problem* (6) *has a unique generalized solution $\psi \in L_2(\Omega)$ such that*

$$\|\psi\|_{L_2(\Omega)} \le M\|\mu\|_{L_2(\mathcal{D})}.$$

**Theorem 5** (existence of a trace). *Let $\psi \in L_2(\Omega)$ be a generalized solution to adjoint problem* (6) *with $\mu \in L_2(\mathcal{D})$. Then there exists a trace $\psi_x(h, y) \in L_2(\mathcal{D})$, and the following estimate holds*:

$$\|\psi_x(h, y)\|_{L_2(\mathcal{D})} \le M\|\mu\|_{L_2(\mathcal{D})}.$$

We introduce the operator $A : q(y) \to u(0, y)$, where $u(x, y)$ is the solution to the direct problem (3).

Then the adjoint operator of $A$ has the form $A^* : \mu(y) \to \psi_x(h, y)$, where $\psi(x, y)$ is the solution to the adjoint problem (6).

It follows from Theorems 2 and 5 that the operators $A$ and $A^*$ act from $L_2(\mathcal{D})$ into $L_2(\mathcal{D})$. Therefore, the inverse problem (3)–(5) can be written down in the operator form

$$Aq = f. \tag{7}$$

We will find the solution of (7) minimizing the data misfit functional [7]

$$J(q) = \|Aq - f\|^2_{L_2(\mathcal{D})}.$$

Let us consider the steepest descent method

$$q_{n+1} = q_n - \alpha_n J'(q_n), \qquad n = 0, 1, 2, \dots,$$

where $J'(q_n)$ is the gradient of the functional and $\alpha_n$ is the descent parameter:

$$J'(q) = 2A^*(Aq - f),$$

$$\alpha_n = \arg\min_{\alpha \geq 0} J(q_n - \alpha J'(q_n)) = \frac{\|J' q_n\|^2}{2\|A(J' q_n)\|^2}$$

Note, that the $J'(q)$ is calculated by

$$J'(q)(x, y) = \psi_x(h, y),$$

where $\psi(x, y)$ is a solution to the adjoint problem (6) with

$$\mu(y) = 2(Aq - f) = 2(u(0, y) - f(y)).$$

**Theorem 6** (the rate of convergence with respect to the functional). *Assume that the problem $Aq = f$ with $f \in L_2(\mathcal{D})$ has a solution $q_* \in L_2(\mathcal{D})$. Then there exists $M > 0$ such that the steepest descent method converges with respect to the functional, and the following estimate holds*:

$$J(q_n) \leq \frac{M}{n}, \quad n = 1, 2, \dots$$

**Theorem 7.** *Assume that the problem $Aq = f$ with $f \in L_2(\mathcal{D})$ has a solution $q_* \in L_2(\mathcal{D})$. Then there exists $M > 0$ such that the sequence $\{u_n\}$ of solutions to direct problems* (3), (4) *for the corresponding iterations $q_n$ converges to the exact solution $u_* \in L_2(\Omega)$ to problem* (1), (2), *and the following estimate holds*:

$$\int_{\mathcal{D}} (u_n(x, y) - u_*(x, y))^2 \, dy \leq M n^{\frac{x-h}{h}}, \quad x \in (0, h).$$

**Theorem 8.** *Assume that the problem $Aq = f$ with $f \in L_2(\mathcal{D})$ has a solution $q_* \in L_2(\mathcal{D})$. Let us have noisy data $\|f - f^\delta\| \le \delta$. Then there exists $M_1 > 0$ and $M_2 > 0$ such that the sequence $\{u_n\}$ of solutions to direct problems* (3), (4) *for the corresponding iterations $u_n^\delta$ converges to the exact solution $u_* \in L_2(\Omega)$ to problem* (1), (2) *and the following estimate holds*:

$$\int_{\mathcal{D}} (u_n^\delta(x,y) - u_*(x,y))^2\,dy \le M_1\delta + M_2 n^{\frac{x-h}{h}}, \quad x \in (0,h).$$

The same theoretical results can be obtained for the Landweber iteration method and the conjugate-gradient method.

## 3. The SVD analysis of the continuation problem

Let us consider the continuation problem for the Helmholtz equation for a simple geometry and homogeneous media:

$$\Delta u + k^2 u = 0, \qquad x \in (0,h), \quad y \in (0,\pi); \tag{8}$$

$$\begin{aligned} &u(0,y) = f(y), && y \in (0,\pi),\\ &u_x(0,y) = 0, && y \in (0,\pi),\\ &u(x,0) = u(x,\pi) = 0, && x \in (0,h). \end{aligned} \tag{9}$$

Here $k^2 = \varepsilon\omega^2 - i\sigma\omega$, $\omega$ is the frequency, $\varepsilon$ and $\sigma$ are positive constants.

Continuation problem (8), (9) consists in finding the function $u(x,y)$ in the domain $x \in (0,h)$, $y \in (0,\pi)$ by the given boundary conditions (9).

Let us formulate the continuation problem in the form of the inverse problem. We introduce the direct problem:

$$\begin{aligned} &\Delta u + k^2 u = 0, && x \in (0,h), \quad y \in (0,\pi),\\ &u_x(0,y) = 0, && y \in (0,\pi),\\ &u(h,y) = q(y), && y \in (0,\pi),\\ &u(x,0) = u(x,\pi) = 0, && x \in (0,h). \end{aligned} \tag{10}$$

The inverse problem: find the function $q(y)$ using additional information

$$u(0,y) = f(y), \qquad y \in (0,\pi). \tag{11}$$

The operator statement of inverse problem (10), (11) can be written down in the form $Aq = f$, where $A : L_2(0,\pi) \to L_2(0,\pi)$ [7].

Let us find the solution to direct problem (10). Let $q(y)$ have the form

$$q(x) = \sum_{m=1}^{\infty} q^{(m)} \sin my,$$

and we find the solution to direct problem

$$u(x,y)=\sum_{m=1}^{\infty}u^{(m)}(x)\sin my$$

solving a sequence of direct problems:

$$u^{(m)}_{xx}+k_m^2u^{(m)}=0,\qquad x\in(0,h), \tag{12}$$

$$u^{(m)}_{x}(0)=0,\qquad u^{(m)}(h)=q^{(m)}. \tag{13}$$

Here $k_m^2=\varepsilon\omega^2-m^2-i\sigma\omega$.

The general solution of equation (12) has the following form:

$$u^{(m)}(x)=C_1e^{\lambda_m x}+C_2e^{-\lambda_m x}.$$

Here $\sqrt{k_m^2}=\pm\lambda_m$, $\lambda_m=\alpha_m+i\beta_m$ and

$$\alpha_m=\sqrt{\frac{\sqrt{(m^2-\varepsilon\omega^2)^2+\sigma^2\omega^2}+m^2-\varepsilon\omega^2}{2}},$$

$$\beta_m=\sqrt{\frac{\sqrt{(m^2-\varepsilon\omega^2)^2+\sigma^2\omega^2}-m^2+\varepsilon\omega^2}{2}}.$$

Therefore, the solution to problem (12), (13) is given by the formula

$$u^{(m)}(x)=\frac{\cosh\lambda_m x}{\cosh\lambda_m h}q^{(m)}.$$

Then the solution to direct problem (10) is given by the Fourier series

$$u(x,y)=\sum_{m=1}^{\infty}\frac{\cosh\lambda_m x}{\cosh\lambda_m h}q^{(m)}\sin my$$

and the solution of inverse problem (10), (11) is

$$q(y)=\sum_{m=1}^{\infty}f^{(m)}\cosh\lambda_m h\cdot\sin my.$$

Thus, singular values of the operator $A$ have the form

$$\sigma_m(A)=\frac{1}{|\cosh\lambda_m h|}=\frac{\sqrt{2}}{\sqrt{\cosh 2\alpha_m h+\cos 2\beta_m h}}.$$

Let us consider some cases of singular values of the operator $A$.

**Example 1.** The Laplace equation—$\varepsilon = 0$, $\sigma = 0$:

$$\sigma_m(A) = \frac{1}{\cosh mh}.$$

**Example 2.** The parabolic equation—$\varepsilon = 0$, $\sigma \neq 0$:

$$\sigma_m(A) = \frac{\sqrt{2}}{\sqrt{\cosh 2\alpha_m h + \cos 2\beta_m h}},$$

$$\alpha_m = \sqrt{\frac{\sqrt{m^4 + \sigma^2\omega^2} + m^2}{2}}, \qquad \beta_m = \sqrt{\frac{\sqrt{m^4 + \sigma^2\omega^2} - m^2}{2}}.$$

**Example 3.** The acoustic equation—$\varepsilon \neq 0$, $\sigma = 0$:

$$\sigma_m(A) = \begin{cases} \dfrac{1}{|\cos k_m h|}, & m^2 \leq \varepsilon\omega^2, \\ \dfrac{1}{\cosh k_m h}, & \varepsilon\omega^2 < m^2, \end{cases}$$

and $k_m^2 = \varepsilon\omega^2 - m^2$.

The singular values strongly depend on the wave number $k_m$ [5]. In the low frequency domain $m^2 \leq \varepsilon\omega^2$ the singular values of $A$ are bounded from below by 1, while for a high-frequency domain, singular values exponentially decay. The most important fact is that in a low-frequency domain, the operator $A$ is continuously invertible and this domain increases with $k_m^2$.

## 4. Numerical experiments

**4.1. Stability analysis.** Let us analyze the ill-posedness of the continuation problem when the data are known on the surface $x = 0$, only.

Let us consider the direct problem

$$\begin{aligned} &\Delta u + k^2 u = 0, \quad x \in (0, h),\ y \in (0, 1), \\ &u_x(0, y) = 0, \quad u(h, y) = q(y), \\ &u(x, 0) = q_1(x), \quad u(x, 1) = q_2(x). \end{aligned}$$

The inverse problem: find the functions $q(y)$, $q_1(x)$ and $q_2(x)$ from the known additional information

$$u(0, y) = f(y).$$

We have fixed $N_x = N_y = 50$. A decay of singular values (Figure 1) shows that the problem becomes more unstable with increasing unknown boundary conditions.

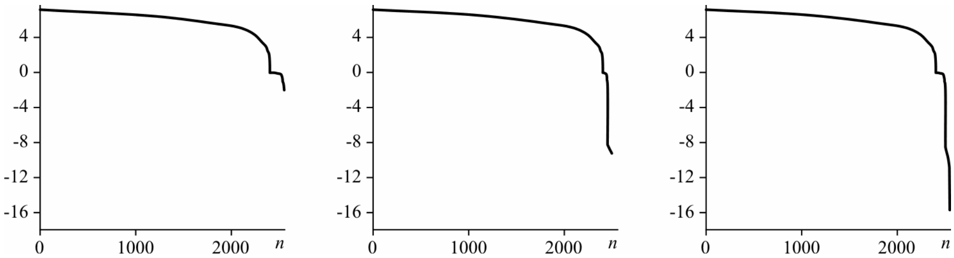


**Figure 1.** The $\log \sigma_n$ function: on the left — $q$ is unknown; in the middle — $q$ and $q_1$ are unknown; and on the right — $q$, $q_1$, and $q_2$ are unknown

**4.2. Solution of continuation problem.** We consider the continuation problem in the domain $x \in [0, 1]$ and $y \in [-3, 3]$. Let $\varepsilon = 1$ in the medium, and there is an inclusion behind the boundary with $\varepsilon = 40$, which is located in $x \in [1.5, 2.5]$ and $y \in [1, 2.5]$ (Figure 2).

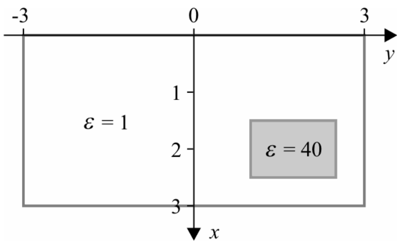


**Figure 2.** Calculation domain

We apply Finite Element Method for the direct and the adjoint problem solutions using the domain triangulation with 427,336 triangles and 214,767 vertices.

In Figure 3a, the inverse problem data are shown: curve 1 is exact data, curve 2 is approximate data after 1500 iterations.

In Figure 3b, the inverse problem solutions are shown: curve 1 is the exact solution, curve 2 is the initial approach, curve 3 is an approximate inverse problem solution after 1500 iterations.

It is evident that the boundary condition $q(y)$ depends on the inclusion outside the domain.

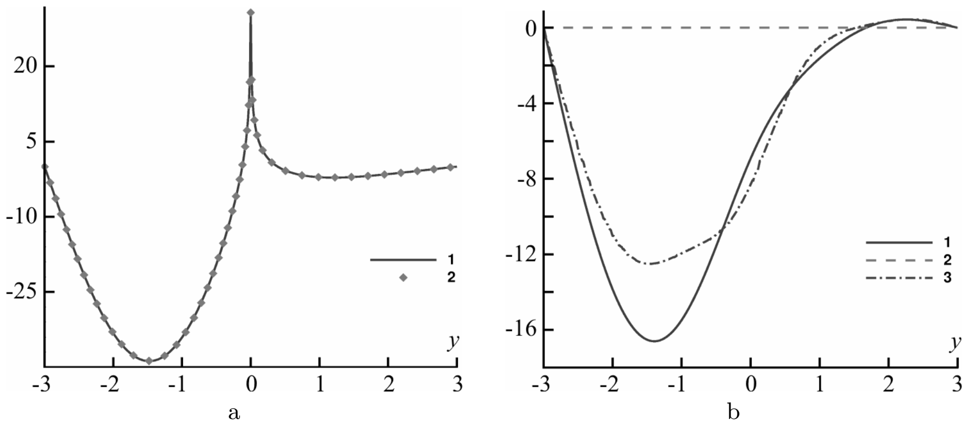


**Figure 3**

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