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Solution of 3D non-stationary problems of impulse electric prospecting

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Abstract. This paper considers some aspects of modeling non-stationary electromagnetic fields for the 3D domains, including inhomogeneous conducting media. The source of such fields is an underground power line. To describe the fields in conducting media, a vector magnetic potential and a scalar electric potential are used, while a magnetic field in non-conducting media is described by a scalar magnetic potential. Systems of equations in conducting media are integrated by the Krank–Nickolson scheme. The conjugation conditions of the vector and the scalar magnetic potentials on the interfaces between the conducting and the non-conducting media hold with mean time steps. The scalar equations are solved by a scalar finite element method of second order, and the vector equations — by the Nedelek vector finite element method of second order of the second kind. The vector variation statements of the problems include the Lagrange factors, governing the divergent properties of vector values.

Introduction

Methods of the impulse electric prospecting are intended for recording processes of generation of electromagnetic fields starting with a certain initial stationary state. Electric parameters of media can be judged from the character of formation of fields. Initial fields are usually generated by the sources, which are essentially smaller than the domain in question. This fact results in generalized formulations of boundary value problems with unrestricted functionals, and one should distinguish peculiarities of the numerical solution. Sources of fields are usually located either on the Earth's surface or in boreholes. Recently, the methods of surveying the sea bottom have gained in importance [1], the source being located directly in the conducting medium, i.e. in the sea water. The domain of electromagnetic fields propagation includes the air and conducting media. This paper is a sequel to the research presented in [2]. Here we consider a quasi-stationary model. The combined use of a scalar magnetic potential in conducting media is discussed in [3]. In this paper, we develop the approach proposed there. Integration with respect to time of a system of equations in conducting media is carried out by the Krank–Nickolson scheme. The time-variation of a scalar magnetic potential is only due to non-stationary processes in conducting media. The scalar magnetic potential is calculated on half-time steps, while the vector potential — on the integer time steps. As compared to [3], a calculation process, conducted in such a manner, brings about decreased computer costs, and the resulting systems of linear algebraic equations possess the properties of symmetry. The scalar equations are solved by the method of scalar finite elements of second order, while the vector equations — by the Nedelek method of vector finite elements of first order of the second kind [4, 5]. The vector variation statements of the problems include the Lagrange factors governing divergent properties of desired vector values. The well-known functional spaces $H^1(\Omega)$, $\boldsymbol{H}(\mathrm{rot},\Omega)$ and their subspaces are employed [6].

1. Statement of the problem

Let us arrange the coordinate system (x, y, z) so that the Earth's surface were in the plane $z = 0$ and the axis z directed (from the ground) into the air. Let the considered domain $\Omega \subset R^3$ be partitioned into the open subdomains Ω_k so that $\bar{\Omega} = \bigcup_k \bar{\Omega}_k$. The subdomains Ω_k are assumed to be homogeneous in terms of their physical properties and characterized by constant values of the electric conductivity σ_k and the permeability μ_k. In what follows, the vector values will be denoted by extra-bold lettering, the scalar values — by standard lettering. Let us consider the air to be a subdomain Ω_0 and the subdomain Ω_1 — to border on the subdomain Ω_0. The air is characterized by the absence of conductivity $\sigma_0 = 0$. Let for all $k \neq 0$, $\sigma_k > 0$. The external boundary Ω is denoted by $\Gamma = \partial\Omega$. Denote the interfaces between the subdomains as $S^{k,l} = \overline{\Omega}_k \cap \overline{\Omega}_l$. The boundary $\Gamma_0 := S^{1,0}$ is the Earth's surface. Let the vector $\boldsymbol{n}$ be the external normal to the boundary Γ, and the vector $\boldsymbol{n}_{k,l}$ — the normal to $S^{k,l}$ directed from the subdomain Ω_k to the subdomain Ω_l.

Let in each subdomain Ω_k, a system of the Maxwell equations with respect to the magnetic intensity $\boldsymbol{H}^k$ and the electric intensity $\boldsymbol{E}^k$ be given. The index k shows that appropriate variables belong to the subdomain Ω_k. In explicit cases, this index is omitted. For non-stationary sources of fields, the system of equations can be written down as

$$\mathrm{rot}\,\boldsymbol{E} + \mu\frac{\partial \boldsymbol{H}}{\partial t} = 0, \qquad \mathrm{div}(\sigma\boldsymbol{E} + \boldsymbol{j}^{\,\mathrm{s}}) = 0, \qquad \mathrm{div}\,\boldsymbol{E}^0 = 0, \tag{1}$$

$$\mathrm{rot}\,\boldsymbol{H} = \sigma\boldsymbol{E} + \boldsymbol{j}^{\,\mathrm{s}}, \qquad \mathrm{div}\,\mu\boldsymbol{H} = 0. \tag{2}$$

On interfaces of the media $S^{k,l}$, the conjugation conditions hold:

$$\boldsymbol{E}^k \times \boldsymbol{n}_{k,l} = \boldsymbol{E}^l \times \boldsymbol{n}_{k,l}, \qquad \sigma_k\boldsymbol{E}^k \cdot \boldsymbol{n}_{k,l} = \sigma_l\boldsymbol{E}^l \cdot \boldsymbol{n}_{k,l}, \tag{3}$$

$$\boldsymbol{H}^k \times \boldsymbol{n}_{k,l} = \boldsymbol{H}^l \times \boldsymbol{n}_{k,l}, \qquad \mu_k\boldsymbol{H}^k \cdot \boldsymbol{n}_{k,l} = \mu_l\boldsymbol{H}^l \cdot \boldsymbol{n}_{k,l}. \tag{4}$$

The external boundary of the domain Γ will be considered to be sufficiently remote from sources, and on it, the Dirichlet conditions of inhomogeneities to be valid (the essence of these conditions will be explained below):

$$\boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{E}_{\mathrm{i}} \times \boldsymbol{n}, \qquad \boldsymbol{H} \times \boldsymbol{n} = \boldsymbol{H}_{\mathrm{i}} \times \boldsymbol{n}. \tag{5}$$

On the Earth's surface Γ_0, the conjugation condition for the electric field normal component turns to the condition:

$$\boldsymbol{E} \cdot \boldsymbol{n}_{1,0} = 0. \tag{6}$$

A non-stationary source of the electric and the magnetic fields is described by the density of the extraneous current $\boldsymbol{j}^{\mathrm{s}}$. Consider the case, when the source is an underground horizontal power line, located in the subdomain Ω_1 on the straight line, parallel to the axis OX. Diameter of the line section falls far short of the size of the domain in question. Let the ends of this line be at the points $A = (x_A, y_A, z_A)$ and $B = (x_B, y_A, z_A)$. At the initial time, the constant current I^{s} provided by the external EMF flows from the point A towards the point B. Then the EMF is switched off. The source current density is expressed as $\boldsymbol{j}^{\mathrm{s}} = (j_x, 0, 0)$, where

$$j_x = I^{\mathrm{s}} \cdot (1 - \theta(t)) \cdot (\theta(x - x_A) - \theta(x - x_B)) \cdot \delta(y - y_A) \cdot \delta(z - z_A),$$

$\theta(t)$ is the Heaviside function, $\delta(t)$ is the Dirac function:

$$\theta(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0, \end{cases} \qquad \delta(t) = \begin{cases} 0, & t \neq 0, \\ +\infty, & t = 0, \end{cases} \qquad \int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

In the sequel, throughout this paper, we will assume that $\mu_k = \mu_0 = \mu =$ const. The solution of the problem contains singularity, associated with a peculiarity in the source:

$$\operatorname{div} \boldsymbol{j}^{\mathrm{s}} = I^{\mathrm{s}} \cdot (1 - \theta(t)) \cdot (\delta(x - x_A) - \delta(x - x_B)) \cdot \delta(y - y_A) \cdot \delta(z - z_A).$$

Therefore, let us present the sought for fields as the sum of primary and anomalous fields:

$$\boldsymbol{E} = \boldsymbol{E}_{\mathrm{i}} + \boldsymbol{E}_{\mathrm{a}}, \qquad \boldsymbol{H} = \boldsymbol{H}_{\mathrm{i}} + \boldsymbol{H}_{\mathrm{a}}.$$

The primary fields satisfy the following boundary value problem, formulated for the conducting homogeneous half-space $z > 0$:

$$\operatorname{rot} \boldsymbol{E}_{\mathrm{i}} + \mu \frac{\partial \boldsymbol{H}_{\mathrm{i}}}{\partial t} = \boldsymbol{0}, \qquad \operatorname{div} \mu \boldsymbol{H}_{\mathrm{i}} = 0 \qquad \text{in } R^3; \tag{7}$$

$$\operatorname{rot} \boldsymbol{H}_{\mathrm{i}} = \sigma_1 \boldsymbol{E} + \boldsymbol{j}^{\mathrm{s}}, \qquad \operatorname{div}(\sigma_1 \boldsymbol{E}_{\mathrm{i}} + \boldsymbol{j}^{\mathrm{s}}) = 0 \qquad \text{for } z < 0; \tag{8}$$

$$\operatorname{rot} \boldsymbol{H}_{\mathrm{i}} = \boldsymbol{0}, \qquad \operatorname{div} \boldsymbol{E}_{\mathrm{i}} = 0 \qquad \text{for } z > 0. \tag{9}$$

On the Earth's surface $z = 0$, the continuity conditions of tangent components of primary fields are valid. The normal component of the primary magnetic field is continuous as well. For the normal component of the primary electric field holds

$$\boldsymbol{E}_{\rm i}\cdot\boldsymbol{n}_{1,0}=0\quad\text{for }\ z=0. \tag{10}$$

On the infinity

$$\boldsymbol{E}_{\rm i}\to\boldsymbol{0},\qquad \boldsymbol{H}_{\rm i}\to\boldsymbol{0}. \tag{11}$$

Problem (7)–(11) is supplemented with values of primary fields at the initial time [2].

A solution to the problem in question is based on the classical solution to the heat conductivity problem, which has been much studied [7]. The solution is analytically obtained in terms of integrals of the Bessel function. Let us next consider that primary fields are vector functions defined at each spatial point and at each instant of time. The values $\boldsymbol{E}_{\rm i}$ and $\boldsymbol{H}_{\rm i}$ on the boundary Γ define the Dirichlet condition (5) for the fields $\boldsymbol{E}$ and $\boldsymbol{H}$.

The presence of anomalous fields is due to the inhomogeneity of conducting media $\Omega^{\rm c}:=\Omega\setminus\overline{\Omega}_0$. Let us formulate the problem for anomalous fields. Further, the air Ω_0 and the conducting media $\Omega^{\rm c}$ will be considered separately. The problem is formulated in the bounded domain Ω like the initial problem:

$$\operatorname{rot}\boldsymbol{E}_{\rm a}+\mu\frac{\partial\boldsymbol{H}_{\rm a}}{\partial t}=\boldsymbol{0},\quad \operatorname{div}\sigma\boldsymbol{E}_{\rm a}=0\qquad\text{in }\ \Omega^{\rm c}, \tag{12}$$

$$\operatorname{rot}\boldsymbol{H}_{\rm a}=\sigma\boldsymbol{E}_{\rm a}+(\sigma-\sigma_1)\boldsymbol{E}_{\rm i}\qquad\text{in }\ \Omega^{\rm c}, \tag{13}$$

$$\operatorname{rot}\boldsymbol{H}_{\rm a}=\boldsymbol{0}\qquad\text{in }\ \Omega_0, \tag{14}$$

$$\operatorname{div}\mu\boldsymbol{H}=0\qquad\text{in }\ \Omega. \tag{15}$$

On the interfaces of the media $S^{k,l}$, the conjugation conditions hold:

$$\boldsymbol{E}^k_{\rm a}\times\boldsymbol{n}_{k,l}=\boldsymbol{E}^l_{\rm a}\times\boldsymbol{n}_{k,l}, \tag{16}$$

$$\sigma_k\boldsymbol{E}^k_{\rm a}\cdot\boldsymbol{n}_{k,l}-\sigma_l\boldsymbol{E}^l_{\rm a}\cdot\boldsymbol{n}_{k,l}=(\sigma_l-\sigma_k)\boldsymbol{E}_{\rm i}\cdot\boldsymbol{n}_{k,l}, \tag{17}$$

$$\boldsymbol{H}^k_{\rm a}\times\boldsymbol{n}_{k,l}=\boldsymbol{H}^l_{\rm a}\times\boldsymbol{n}_{k,l},\qquad \boldsymbol{H}^k_{\rm a}\cdot\boldsymbol{n}_{k,l}=\boldsymbol{H}^l_{\rm a}\cdot\boldsymbol{n}_{k,l}. \tag{18}$$

On the Earth's surface Γ_0, owing to (10), conjugation condition (17) turns to:

$$\sigma_1\boldsymbol{E}_{\rm a}\cdot\boldsymbol{n}_{1,0}=0. \tag{19}$$

On the external boundary of the domain Γ, the Dirichlet homogeneous conditions hold:

$$\boldsymbol{E}_{\rm a}\times\boldsymbol{n}=\boldsymbol{0},\qquad \boldsymbol{H}_{\rm a}\times\boldsymbol{n}=\boldsymbol{0}. \tag{20}$$

The initial data for anomalous fields are obtained from this system of equations at the initial instant [2]. Let us next consider that the initial data are known:

$$\boldsymbol{E}_{\rm a}\big|_{t=0}=\boldsymbol{E}^0_{\rm a},\qquad \boldsymbol{H}_{\rm a}\big|_{t=0}=\boldsymbol{H}^0_{\rm a}. \tag{21}$$

2. Variation statements of the problems for anomalous fields in terms of potentials

Similar to [2], to describe anomalous fields, we introduce a scalar magnetic potential Φ into Ω_0 and a vector magnetic potential $\mathbf{A}$ into the conducting media Ω^{c} with appropriate calibration condition:

$$\boldsymbol{H}_{\mathrm{a}} = -\nabla\Phi \quad \text{in } \Omega_0,$$
$$\boldsymbol{H}_{\mathrm{a}} = \frac{1}{\mu}\operatorname{rot}\boldsymbol{A}, \quad \operatorname{div}\sigma\boldsymbol{A} = 0 \quad \text{in } \Omega^{\mathrm{c}}.$$

In this case, the intensity of the anomalous electric field $\boldsymbol{E}_{\mathrm{a}}$ in the subdomain Ω^{c} can be presented in the form $\boldsymbol{E}_{\mathrm{a}} = -\dfrac{\partial\boldsymbol{A}}{\partial t} - \nabla U$. Let us rewrite equations (12)–(21) in terms of the potentials:

$$-\operatorname{div}\mu\nabla\Phi = 0 \qquad \text{in } \Omega_0, \tag{22}$$
$$\Phi = 0 \qquad \text{on } \Gamma\cap\overline{\Omega}_0, \tag{23}$$
$$\operatorname{rot}\frac{1}{\mu}\operatorname{rot}\boldsymbol{A} + \sigma\frac{\partial\boldsymbol{A}}{\partial t} + \sigma\nabla U = (\sigma-\sigma_1)\boldsymbol{E}_{\mathrm{i}} \qquad \text{in } \Omega^{\mathrm{c}}, \tag{24}$$
$$\operatorname{div}\sigma\boldsymbol{A} = 0 \qquad \text{in } \Omega^{\mathrm{c}}, \tag{25}$$
$$\boldsymbol{A}\times\boldsymbol{n} = \mathbf{0} \qquad \text{on } \Gamma\cap\overline{\Omega}^{\mathrm{c}} \tag{26}$$

and the conjugation conditions on the interfaces between the media:

$$-\mu\nabla\Phi\cdot\boldsymbol{n}_{1,0} = \operatorname{rot}\boldsymbol{A}^1\cdot\boldsymbol{n}_{1,0} \qquad \text{on } \Gamma_0, \tag{27}$$
$$-\nabla\Phi\times\boldsymbol{n}_{1,0} = \frac{1}{\mu}\operatorname{rot}\boldsymbol{A}^1\times\boldsymbol{n}_{1,0} \qquad \text{on } \Gamma_0, \tag{28}$$
$$\frac{1}{\mu}\operatorname{rot}\boldsymbol{A}^k\times\boldsymbol{n}_{k,l} = \frac{1}{\mu}\operatorname{rot}\boldsymbol{A}^l\times\boldsymbol{n}_{k,l} \qquad \text{on } S^{k,l}, \tag{29}$$
$$\sigma_k\boldsymbol{A}^k\cdot\boldsymbol{n}_{k,l} = \sigma_l\boldsymbol{A}^l\cdot\boldsymbol{n}_{k,l} \qquad \text{on } S^{k,l}. \tag{30}$$

Let us consider the conjugation conditions for the potential U on the interface between the media $S^{k,l}$. For this, let us take a divergence of (24) and consider the result of its application on the surfaces $S^{k,l}$. Using the representation of $\boldsymbol{E}_{\mathrm{a}}$ by potentials and relations (17), (30), we obtain

$$\sigma_k\nabla U^k\cdot\boldsymbol{n}_{k,l} - \sigma_l\nabla U^l\cdot\boldsymbol{n}_{k,l} = (\sigma_k-\sigma_l)\boldsymbol{E}_{\mathrm{i}}\cdot\boldsymbol{n}_{k,l}. \tag{31}$$

On the Earth's surface Γ_0, condition (31) turns to the following:

$$\sigma_1\nabla U^1\cdot\boldsymbol{n}_{1,0} = 0. \tag{32}$$

From (24)–(32), it is possible to obtain a closed formulation of the problem for the electric potential U:

$$\operatorname{div}(\sigma\nabla U) = 0 \qquad \text{in } \Omega^{\mathrm{c}}, \tag{33}$$

$$\sigma_1\nabla U^1 \cdot \boldsymbol{n}_{1,0} = 0 \qquad \text{on } \Gamma_0, \tag{34}$$

$$\sigma_k\nabla U^k \cdot \boldsymbol{n}_{k,l} - \sigma_l\nabla U^l \cdot \boldsymbol{n}_{k,l} = (\sigma_k - \sigma_l)\boldsymbol{E}_{\mathrm{i}} \cdot \boldsymbol{n}_{k,l} \qquad \text{on } S^{k,l}, \tag{35}$$

$$U = 0 \qquad \text{on } \Gamma \cap \overline{\Omega}^{\mathrm{c}}. \tag{36}$$

The electric potential U can be calculated at each instant and depends only on the current values of the primary electric field $\boldsymbol{E}_{\mathrm{i}}$ on the interface between the conducting media $S^{k,l}$. Let us rearrange the term with the potential to the right-hand side of equation (24) and introduce the Lagrange factors P into the conducting subdomain Ω^{c}:

$$\operatorname{rot}\frac{1}{\mu}\operatorname{rot}\boldsymbol{A} + \sigma\frac{\partial \boldsymbol{A}}{\partial t} - \sigma\nabla P = (\sigma - \sigma_1)\boldsymbol{E}_{\mathrm{i}} - \sigma\nabla U \qquad \text{in } \Omega^{\mathrm{c}}. \tag{37}$$

Let us provide the problem with boundary conditions for the Lagrange factors:

$$\nabla P \cdot \boldsymbol{n}_{1,0} = 0 \qquad \text{on } \Gamma_0, \tag{38}$$

$$P = 0 \qquad \text{on } \Gamma \cap \overline{\Omega}^{\mathrm{c}}. \tag{39}$$

Consider the initial data to be known [2]:

$$\boldsymbol{A}\big|_{t=0} = \boldsymbol{A}^0, \qquad \Phi\big|_{t=0} = \Phi^0.$$

Now, formulate the generalized statements of the problems. The problem for the magnetic scalar potential Φ is solved with the help of a subspace of the functions $H_\Phi \subset H^1(\Omega_0)$ with zero trace on the boundary $\Gamma \cap \overline{\Omega}_0$. For the vector magnetic potential $\boldsymbol{A}$ we introduce a subspace of the functions $\boldsymbol{H}_A \subset \boldsymbol{H}(\operatorname{rot}, \Omega^{\mathrm{c}})$ with zero tangent trace on the boundary $\Gamma \cap \overline{\Omega}^{\mathrm{c}}$. For the Lagrange factors and the scalar electric potential we introduce a subspace $H_U \subset H^1(\Omega^{\mathrm{c}})$ of the functions with zero trace on the boundary $\Gamma \cap \overline{\Omega}^{\mathrm{c}}$.

At first, we formulate a generalized problem appropriate to problem (33)–(36) for the scalar potential U:

Find a function $U \in C^0((0,T); H_U)$ *such that* $\forall t \in (0,T)$ *and* $\forall V \in H_U$

$$\int_{\Omega^{\mathrm{c}}} \sigma\nabla U \cdot \nabla V\, d\Omega = \sum_{(k,l)\neq(1,0)} \int_{S^{k,l}} V\,(\sigma_k - \sigma_l)\boldsymbol{E}_{\mathrm{i}} \cdot \boldsymbol{n}_{k,l}\, dS. \tag{40}$$

Now we formulate the problems for magnetic potentials and the Lagrange factor appropriate to problem (25)–(30), (37)–(39) in assumption that the scalar electric potential U is the function defined from (40):

Find functions

$$\Phi \in C^0((0,T); H_\Phi), \quad (\boldsymbol{A}, P) \in C^1(0,T); \boldsymbol{H}_A) \times C^0((0,T); H_U)$$

such that $\forall t \in (0,T)$ *and* $\forall W \in H_\Phi$, $\forall(\boldsymbol{B}, V) \in \boldsymbol{H}_A \times H_U$

$$\int_{\Omega_0} \mu \nabla\Phi \cdot \nabla W \, d\Omega = -\int_{\Gamma_0} W \operatorname{rot} \boldsymbol{A} \cdot \boldsymbol{n}_{1,0} \, dS,$$

$$\int_{\Omega^c} \frac{1}{\mu} \operatorname{rot} \boldsymbol{A} \cdot \operatorname{rot} \boldsymbol{B} \, d\Omega + \frac{\partial}{\partial t} \int_{\Omega^c} \sigma \boldsymbol{A} \cdot \boldsymbol{B} \, d\Omega - \int_{\Omega^c} \sigma \nabla P \cdot \boldsymbol{B} \, d\Omega$$
$$= \int_{\Omega^c} (\sigma - \sigma_1) \boldsymbol{E}_{\mathrm{i}} \cdot \boldsymbol{B} \, d\Omega - \int_{\Omega^c} \sigma \nabla U \cdot \boldsymbol{B} \, d\Omega - \int_{\Gamma_0} \nabla\Phi \times \boldsymbol{n}_{1,0} \cdot \boldsymbol{B} \, dS,$$
$$-\int_{\Omega^c} \sigma \boldsymbol{A} \cdot \nabla V \, d\Omega = 0.$$

3. Numerical implementation

Let there be a regular family of triangulations T^h of the domain Ω for which the quasi-uniformity condition holds. Denote by S_h and F_h the spaces of finite elements of the second order, which are conformal in the spaces $H^1(\Omega^c)$ and $H^1(\Omega_0)$, respectively. Denote by V_h the space of Nedelek elements of the first order and the second kind, which are conformal in the space $\boldsymbol{H}(\operatorname{rot}, \Omega^c)$. The degrees of freedom of functions from the spaces S_h and F_h are associated with function values at the nodes and the middles of the triangulation edges, and for functions from the space $\boldsymbol{V}_h$—with moments of vector functions on the triangulation edges. Introduce the spaces $Q_h = F_h \cap H_\Phi$, $\boldsymbol{X}_h = \boldsymbol{V}_h \cap \boldsymbol{H}_A$, and $Y_h = S_h \cap H_U$. It is possible to formulate finite-dimensional (according to spatial variables) analogues of projective problems for anomalous fields.

Find a function $U_h \in C^0((0,T); Y_h)$ *such that* $\forall t \in (0,T)$ *and* $\forall V_h \in Y_h$

$$\int_{\Omega^c} \sigma \nabla U_h \cdot \nabla V_h \, d\Omega = \sum_{(k,l) \neq (1,0)} \int_{S^{k,l}} V_h \, (\sigma_k - \sigma_l) \boldsymbol{E}_{\mathrm{i}} \cdot \boldsymbol{n}_{k,l} \, dS.$$

Find functions

$$\Phi_h \in C^0((0,T); Q_h), \quad (\boldsymbol{A}_h, P_h) \in C^1((0,T); \boldsymbol{X}_h) \times C^0((0,T); Y_h)$$

such that $\forall t \in (0,T)$ *and* $\forall W_h \in Q_h$, $\forall(\boldsymbol{B}_h, V_h) \in \boldsymbol{X}_h \times Y_h$

$$\int_{\Omega_0} \mu \nabla\Phi_h \cdot \nabla W_h \, d\Omega = -\int_{\Gamma_0} W_h \operatorname{rot} \boldsymbol{A}_h \cdot \boldsymbol{n}_{1,0} \, dS,$$

$$\int_{\Omega^c} \frac{1}{\mu} \operatorname{rot} \boldsymbol{A}_h \cdot \operatorname{rot} \boldsymbol{B}_h \, d\Omega + \frac{\partial}{\partial t} \int_{\Omega^c} \sigma \boldsymbol{A}_h \cdot \boldsymbol{B}_h \, d\Omega - \int_{\Omega^c} \sigma \nabla P_h \cdot \boldsymbol{B}_h \, d\Omega$$
$$= \int_{\Omega^c} (\sigma - \sigma_1) \boldsymbol{E}_{\mathrm{i}} \cdot \boldsymbol{B}_h \, d\Omega - \int_{\Omega^c} \sigma \nabla U_h \cdot \boldsymbol{B}_h \, d\Omega - \int_{\Gamma_0} \nabla \Phi_h \times \boldsymbol{n}_{1,0} \cdot \boldsymbol{B} \, dS,$$
$$- \int_{\Omega^c} \sigma \boldsymbol{A}_h \cdot \nabla V_h \, d\Omega = 0.$$

We now turn our attention to the linear algebraic equations system (SLAE). To do this, let us introduce the bases of the spaces Q_h, $\boldsymbol{X}_h$ and Y_h:

$$Q_h = \operatorname{span}\{\varphi_k;\ k = 1, \ldots, N_Q\}, \quad \boldsymbol{X}_h = \operatorname{span}\{\boldsymbol{N}_i;\ i = 1, \ldots, N_X\},$$
$$Y_h = \operatorname{span}\{\psi_k;\ k = 1, \ldots, N_Y\}.$$

The desired functions $U_h \in C^0((0,T); Y_h)$, $\Phi_h \in C^0((0,T); Q_h)$, $A_h \in C^1((0,T); X_h)$, $P_h \in C^0((0,T); Y_h)$ can be presented as

$$U_h = \sum_{l=1}^{N_Y} u_l \psi_l, \quad \Phi_h = \sum_{k=1}^{N_Q} f_k \varphi_k, \quad \boldsymbol{A}_h = \sum_{j=1}^{N_X} a_j \boldsymbol{N}_j, \quad P_h = \sum_{l=1}^{N_Y} p_l \psi_l.$$

The finite-dimensional projective problems are written down as SLAE with respect to unknown expansion coefficients that depend on time:

$$\mathcal{L}\boldsymbol{u} = \boldsymbol{e}, \qquad \mathcal{A}\boldsymbol{a} + \frac{\partial}{\partial t}\mathcal{M}\boldsymbol{a} - \mathcal{B}^T \boldsymbol{p} = \boldsymbol{F}(\boldsymbol{f}, \boldsymbol{u}),$$
$$-\mathcal{B}\boldsymbol{a} = \boldsymbol{0}, \qquad \mathcal{C}\boldsymbol{f} = \boldsymbol{G}(\boldsymbol{a}).$$

Here the following matrix-vector notations are used:

$$\mathcal{L} = \left\{ \int_{\Omega_0} \sigma \nabla \psi_k \cdot \nabla \psi_j \, d\Omega;\ k, j = 1, \ldots, N_Y \right\}, \qquad \boldsymbol{u} = (u_1, \ldots, u_{N_Y})^T,$$
$$\boldsymbol{e} = \left\{ \sum_{(k,l) \neq (1,0)} \int_{S^{k,l}} \psi_j (\sigma_k - \sigma_l) \boldsymbol{E}_{\mathrm{i}} \cdot \boldsymbol{n}_{k,l} \, dS;\ j = 1, \ldots, N_Y \right\}^T,$$
$$\mathcal{A} = \left\{ \int_{\Omega^c} \frac{1}{\mu} \operatorname{rot} \boldsymbol{N}_i \cdot \operatorname{rot} \boldsymbol{N}_j \, d\Omega;\ i, j = 1, \ldots, N_X \right\}, \qquad \boldsymbol{a} = (a_1, \ldots, a_{N_X})^T,$$
$$\mathcal{B} = \left\{ \int_{\Omega^c} \sigma \nabla \psi_k \cdot \boldsymbol{N}_i \, d\Omega;\ k = 1, \ldots, N_Y,\ i = 1, \ldots, N_X \right\},$$
$$\mathcal{C} = \left\{ \int_{\Omega_0} \mu \nabla \varphi_k \cdot \nabla \varphi_i \, d\Omega;\ k, i = 1, \ldots, N_Q \right\}, \qquad \boldsymbol{f} = (f_1, \ldots, f_{N_Q})^T,$$
$$\mathcal{M} = \left\{ \int_{\Omega^c} \sigma \boldsymbol{N}_i \cdot \boldsymbol{N}_j \, d\Omega;\ i, j = 1, \ldots, N_X \right\}, \qquad \boldsymbol{p} = (p_1, \ldots, p_{N_Y})^T,$$

$$\boldsymbol{F}(\boldsymbol{f},\boldsymbol{u}) = \left\{ \int_{\Omega^c} (\sigma - \sigma_1)\boldsymbol{E}_{\mathrm{i}} \cdot \boldsymbol{N}_j \, d\Omega - \sum_{l=1}^{N_Y} u_l \int_{\Omega^c} \sigma \nabla \psi_l \cdot \boldsymbol{N}_j \, d\Omega - \right.$$
$$\left. \sum_{k=1}^{N_Q} f_k \int_{\Gamma_0} \nabla \varphi_k \times \boldsymbol{n}_{1,0} \cdot \boldsymbol{N}_j \, dS; \;\; j = 1, \ldots, N_X \right\}^T,$$
$$\boldsymbol{G}(\boldsymbol{a}) = \left\{ - \sum_{j=1}^{N_X} a_j \int_{\Gamma_0} \varphi_k \, \mathrm{rot}\, \boldsymbol{N}_j \cdot \boldsymbol{n}_{1,0} \, dS; \;\; k = 1, \ldots, N_Q \right\}^T.$$

Introduce the time grid $\{t_n, \; n = 0, 1, \ldots\}$. We admit that this grid can be non-uniform. Denote the steps $\tau_n = t_n - t_{n-1}$ and the middles of intervals $t_{n-1/2} = (t_n + t_{n-1})/2$, $n = 1, 2, \ldots$ The superscripts of the sought for vectors $\boldsymbol{a}^n$, $\boldsymbol{f}^{n-1/2}$ will be appropriate for a time step number. The vectors $\boldsymbol{a}^0$ and $\boldsymbol{f}^0$ are assumed to be set by means of the initial conditions.

The scalar magnetic potential $\boldsymbol{f}^{1/2}$ in a non-conducting medium can be calculated by the following scheme:

$$\begin{aligned}
&\mathcal{L}\boldsymbol{u}^0 = \boldsymbol{e}^0,\\
&\mathcal{A}\frac{\boldsymbol{a}^1 + \boldsymbol{a}^0}{2} + \mathcal{M}\frac{\boldsymbol{a}^1 - \boldsymbol{a}^0}{\tau_1} - \mathcal{B}^T\boldsymbol{p}^{1/2} = \boldsymbol{F}(\boldsymbol{f}^0, \boldsymbol{u}^0),\\
&-\mathcal{B}\boldsymbol{a}^1 = 0,\\
&\mathcal{C}\boldsymbol{f}^1 = \boldsymbol{G}(\boldsymbol{a}^1), \quad \boldsymbol{f}^{1/2} = \frac{\boldsymbol{f}^1 + \boldsymbol{f}^0}{2}.
\end{aligned}$$

Then we use a regular Krank–Nikolson scheme, starting from the step $n = 1$:

$$\begin{aligned}
&\mathcal{L}\boldsymbol{u}^{n-1/2} = \boldsymbol{e}^{n-1/2},\\
&\mathcal{A}\frac{\boldsymbol{a}^n + \boldsymbol{a}^{n-1}}{2} + \mathcal{M}\frac{\boldsymbol{a}^n - \boldsymbol{a}^{n-1}}{\tau_n} - \mathcal{B}^T\boldsymbol{p}^{n-1/2} = \boldsymbol{F}(\boldsymbol{f}^{n-1/2}, \boldsymbol{u}^{n-1/2}),\\
&-\mathcal{B}\boldsymbol{a}^n = 0,\\
&\mathcal{C}(\alpha_n \boldsymbol{f}^{n+1/2} + \beta_n \boldsymbol{f}^{n-1/2}) = \boldsymbol{G}(\boldsymbol{a}^n).
\end{aligned}$$

The coefficients are dependent on the time steps:

$$\alpha_n = \frac{\tau_n}{\tau_n + \tau_{n+1}}, \qquad \beta_n = 1 - \alpha_n.$$

Thus, at each iteration we solve a system of equations with respect to a vector magnetic potential and a scalar electric potential in a conducting medium as well as a system of equations for a scalar magnetic potential in the air.

4. Example of numerical calculation

A large share of effort goes into the calculation of a vector magnetic potential and the Lagrange factor in a conducting medium. Let us give an example, characteristic of sensitivity of the algorithm to the presence of inhomogeneities in a conducting medium. For this purpose, consider a domain made from the two horizontal conducting layers with conductivities $\sigma_1 = 3.2$ S/m and $\sigma_2 = 0.5$ S/m. The thickness of each layer equals 3000 m. The lower medium has an inclusion in the shape of $250\times1000\times500$ m parallelepiped with a contrasting conductivity $\sigma_3 = 0.01$ S/m. The horizontal dimensions of the layers make up 6000×6000 m. The common size of the calculation domain is selected so as errors in the boundary conditions on the external boundaries be of minor influence on the analysis of fields inside the domain. The source is in the upper medium at a height of 500 m from the second layer, the length of the power line being 500 m. The current

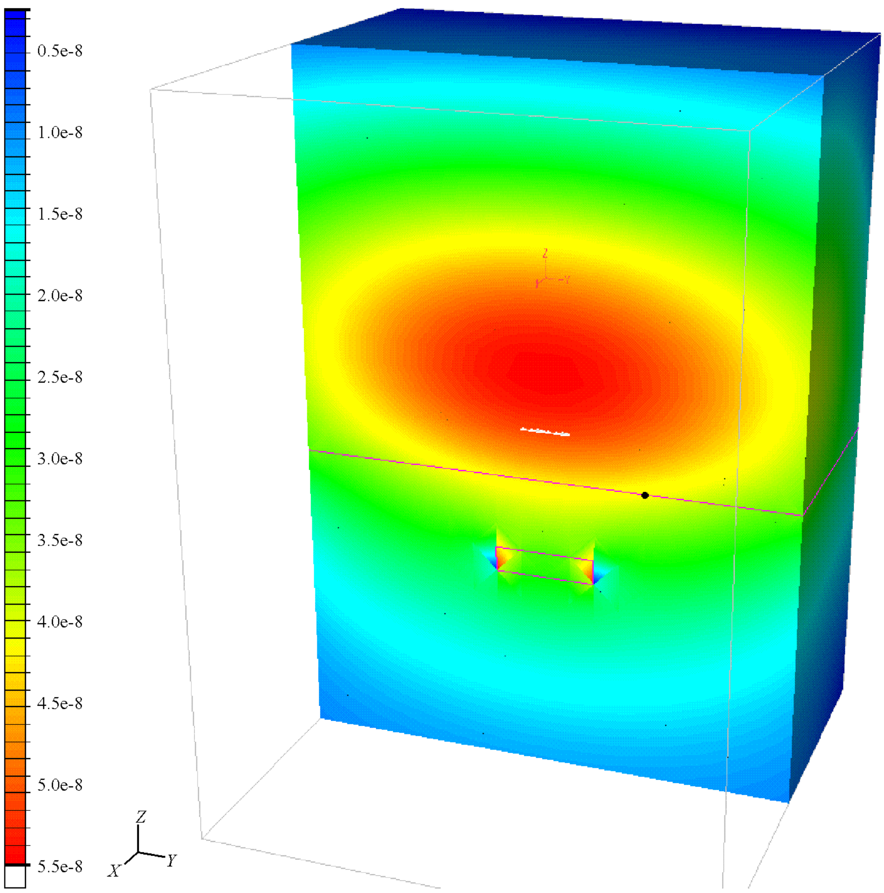

Figure 1

of the source is $I^{\mathrm{s}} = 50$ A. Triangulation of the calculation domain covers 70,000 edges and 10,000 vertices. A relative accuracy of the solution to the SLAE makes up $\varepsilon = 10^{-10}$. We use a non-uniform time grid. The total time interval of integration of a system of equations is $T = 100$ s, the total number of time steps being about 200. The electric field is measured at the interface between two conducting layers in the vertical plane of a source, at a distance of 1000 m from the projection of the power line center onto the interface.

Figure 1 shows y-component of the electric field at an instant $t = 10$ s. The domain is presented as a vertical cross-section by the plane through the source. The solid lines denote the interfaces of conducting media. The source is shown as a white segment. The receiver is shown by a black dot on the interface. The location of axes in space is shown in the left bottom part of the figure. The violation of homogeneity of the electric field on the boundary of a subdomain with a contrasting conductivity is well distinguished.

Figure 2 presents graphs of dependence of y-component of the electric field on time at a measurement point. The solid line corresponds to the case in question. The dotted line corresponds to the case without a contrasting inclusion: $\sigma_3 = \sigma_2$. At the initial time, the level of a signal at the measurements point is almost unvarying, and a relative difference between the curves makes up about 6.5 %. Then the level of the signal falls. At the instant $t = 85$ s, the level of the signals falls by three orders.

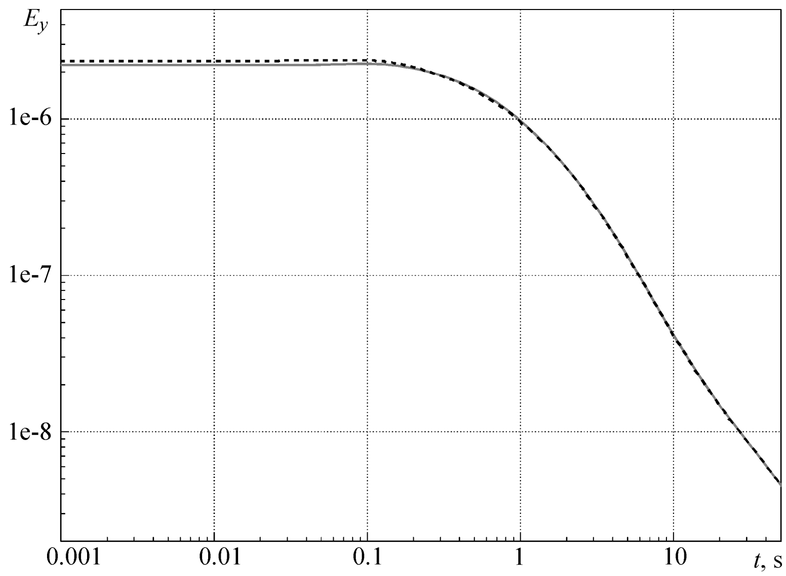

Figure 2

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