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The fundamental solution of the stationary two-velocity hydrodynamics equation with one pressure*

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Abstract. The fundamental solution to describe the three-dimensional steady-state flows of viscous fluids of the two-velocity continuum with pressure phase equilibrium has been obtained.

Keywords: Two-velocity hydrodynamics, viscous fluid, fundamental solution, potentials of single and double layer.

1. Introduction

The study of physical and technical processes in the continuum mechanics begins with constructing a mathematical model. The presence in the upper mantle of partial melts has acquire an important role in geophysical literature. The assumption of formation of a partial melt by first order phase transition has allowed V.N. Dorovsky to explain localization in space of large masses of this substance in dynamic conditions [1]. In this case, the effects of volume magma generation were not taken into account. The account of magma generation in terms of the shear strain mantle thickness was considered in [2], where a continuous medium in the geological time scale is a viscous "fluid-1" due to intrinsic viscosity, or for other reasons, it attacks the necessary thermodynamic conditions of the phase transition. Along the grain boundaries and the inter-grain there begins the accumulation of magma, i.e., "fluid-2" with a viscosity intrinsic of melts known in geology.

Such a melt is integrated in the process of the combined heat and mass transfer and filtered through the system that has generated it. In other words, this theory represents heat and mass transfer dynamics of mutual penetration of one fluid that is less viscous through a medium of greater viscosity viscous as a kind of filtering process. Or by analogy with the Navier–Stokes equations this theory can be called a two-speed system of the Navier–Stokes equations or a two-velocity hydrodynamics.

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The study of viscous compressible / incompressible fluids based on the solution to a complete system of equations of the two-velocity hydrodynamics is promising. In the literature, quite a limited number of cases, allowing the analytical integration of the Navier–Stokes equations are known [3, 4]. The objective of this paper is to construct the fundamental solution for the stationary system of equations of the two-velocity hydrodynamics with pressure phase equilibrium. Such a solution may be useful for testing numerical methods solving the two-velocity hydrodynamics equations.

2. Two-velocity hydrodynamics equations with one pressure

In [5, 6], a nonlinear two-velocity model of fluid motion through a deformable porous medium was developed on the basis of conservation laws, invariance of the equations as related to the Galilei transformations and conditions of thermodynamic consistency. The two-velocity two-fluid hydrodynamic theory of the pressure equilibrium condition of subsystems was constructed in [2]. The equations of motion of a two-velocity medium in the dissipative case with one pressure, in the isothermal case has the form [2]:

$$\frac{\partial \bar{\rho}}{\partial t} + \operatorname{div}(\tilde{\rho}\tilde{\boldsymbol{v}} + \rho\boldsymbol{v}) = 0, \qquad \frac{\partial \tilde{\rho}}{\partial t} + \operatorname{div}(\tilde{\rho}\tilde{\boldsymbol{v}}) = 0, \tag{1}$$

$$\bar{\rho}\Big(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v},\nabla)\boldsymbol{v}\Big) = -\nabla p + \nu\Delta\boldsymbol{v} + (\nu/3+\mu)\nabla\operatorname{div}\boldsymbol{v} + \frac{\tilde{\rho}}{2}\nabla(\tilde{\boldsymbol{v}}-\boldsymbol{v})^2 + \bar{\rho}\boldsymbol{f}, \tag{2}$$

$$\bar{\rho}\Big(\frac{\partial \tilde{\boldsymbol{v}}}{\partial t} + (\tilde{\boldsymbol{v}},\nabla)\tilde{\boldsymbol{v}}\Big) = -\nabla p + \tilde{\nu}\Delta\tilde{\boldsymbol{v}} + (\tilde{\nu}/3+\tilde{\mu})\nabla\operatorname{div}\tilde{\boldsymbol{v}} - \frac{\rho}{2}\nabla(\tilde{\boldsymbol{v}}-\boldsymbol{v})^2 + \bar{\rho}\boldsymbol{f}, \tag{3}$$

where $\tilde{\boldsymbol{v}}$ and $\boldsymbol{v}$ are the velocity vectors of the subsystems that make up the two-velocity continuum with the corresponding partial densities $\tilde{\rho}$ and ρ, ν (μ), and $\tilde{\nu}$ ($\tilde{\mu}$) are the corresponding shear (bulk) viscosities, $\bar{\rho} = \tilde{\rho} + \rho$ is the common density of the two-velocity continuum; $p = p(\bar{\rho}, (\tilde{\boldsymbol{v}}-\boldsymbol{v})^2)$ is a two-velocity equation of state; $\boldsymbol{f}$ is the force mass vector per mass unit.

Rewrite equations (2) and (3) in the equivalent form

$$\begin{aligned}\bar{\rho}\Big(\frac{\partial \boldsymbol{v}}{\partial t} + \frac{1}{2}\nabla(v^2) - \boldsymbol{v}\times\operatorname{rot}\boldsymbol{v}\Big)\\ = -\nabla p + \nu\Delta\boldsymbol{v} + (\nu/3+\mu)\nabla\operatorname{div}\boldsymbol{v} + \frac{\tilde{\rho}}{2}\nabla(\tilde{\boldsymbol{v}}-\boldsymbol{v})^2 + \bar{\rho}\boldsymbol{f},\end{aligned} \tag{4}$$

$$\begin{aligned}\bar{\rho}\Big(\frac{\partial \tilde{\boldsymbol{v}}}{\partial t} + \frac{1}{2}\nabla(\tilde{v}^2) - \tilde{\boldsymbol{v}}\times\operatorname{rot}\tilde{\boldsymbol{v}}\Big)\\ = -\nabla p + \tilde{\nu}\Delta\tilde{\boldsymbol{v}} + (\tilde{\nu}/3+\tilde{\mu})\nabla\operatorname{div}\tilde{\boldsymbol{v}} - \frac{\rho}{2}\nabla(\tilde{\boldsymbol{v}}-\boldsymbol{v})^2 + \bar{\rho}\boldsymbol{f},\end{aligned} \tag{5}$$

From these equations we can derive other ones to determine a change in the vortices in the course of time. For this let us apply the operator rot to both sides of equations (4) and (5). As a result, obtain

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} - \operatorname{rot}\Big(\boldsymbol{v} \times \boldsymbol{\Omega}\Big) = -\operatorname{rot}\Big(\frac{\nabla p}{\bar{\rho}}\Big) + \nu\Delta\boldsymbol{\Omega} + \operatorname{rot}\Big(\frac{\nu/3+\mu}{\bar{\rho}}\nabla \operatorname{div}\boldsymbol{v}\Big) + \operatorname{rot}\Big(\frac{\tilde{\rho}}{2\bar{\rho}}\nabla(\tilde{\boldsymbol{v}}-\boldsymbol{v})^2\Big) + \operatorname{rot}\boldsymbol{f},$$

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} - \operatorname{rot}\Big(\tilde{\boldsymbol{v}} \times \boldsymbol{\Omega}\Big) = -\operatorname{rot}\Big(\frac{\nabla p}{\bar{\rho}}\Big) + \tilde{\nu}\Delta\boldsymbol{\Omega} + \operatorname{rot}\Big(\frac{\tilde{\nu}/3+\tilde{\mu}}{\bar{\rho}}\nabla \operatorname{div}\tilde{\boldsymbol{v}}\Big) - \operatorname{rot}\Big(\frac{\rho}{2\bar{\rho}}\nabla(\tilde{\boldsymbol{v}}-\boldsymbol{v})^2\Big) + \operatorname{rot}\boldsymbol{f}.$$

3. The linear system of equations of two-velocity hydrodynamics of a compressible medium

Without the mass forces $\boldsymbol{f} = 0$, the system of equations (1)–(3) has the solution $\boldsymbol{v} = 0$, $\tilde{\boldsymbol{v}} = 0$, $\rho = \rho^0$, $\tilde{\rho} = \tilde{\rho}^0$ for interphase mixture fluids with, that is at the rest, the uniform pressure $p = p^0$, the partial densities ρ^0, $\tilde{\rho}^0$ and the temperature T.

Let us linearize equations (2), (3) with respect to the hydrodynamic background $\boldsymbol{v} = 0$, $\tilde{\boldsymbol{v}} = 0$, $\rho = \rho^0$, $\tilde{\rho} = \tilde{\rho}^0$, $p = p^0$, i.e.:

$$\boldsymbol{v} = \boldsymbol{v}^1, \quad \tilde{\boldsymbol{v}} = \tilde{\boldsymbol{v}}^1, \quad \rho = \rho^0 + \rho^1, \quad \tilde{\rho} = \tilde{\rho}^0 + \tilde{\rho}^1, \quad p = p^0 + p^1.$$

Substituting these expressions in (1)–(3) and, for brevity, using the notations $\boldsymbol{v}$, $\tilde{\boldsymbol{v}}$, ρ, $\tilde{\rho}$ instead of $\boldsymbol{v}^1$, $\tilde{\boldsymbol{v}}^1$, ρ^1, $\tilde{\rho}^1$, we obtain

$$\frac{\partial \rho}{\partial t} + \rho^0 \operatorname{div}\boldsymbol{v} = 0, \qquad \frac{\partial \tilde{\rho}}{\partial t} + \tilde{\rho}^0 \operatorname{div}\tilde{\boldsymbol{v}} = 0, \tag{6}$$

$$\bar{\rho}^0\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla p + \nu\Delta\boldsymbol{v} + (\nu/3+\mu)\nabla\operatorname{div}\boldsymbol{v} + \bar{\rho}^0\boldsymbol{f}, \tag{7}$$

$$\bar{\rho}^0\frac{\partial \tilde{\boldsymbol{v}}}{\partial t} = -\nabla p + \tilde{\nu}\Delta\tilde{\boldsymbol{v}} + (\tilde{\nu}/3+\tilde{\mu})\nabla\operatorname{div}\tilde{\boldsymbol{v}} + \bar{\rho}^0\boldsymbol{f}. \tag{8}$$

4. The linear steady-state system of two-velocity hydrodynamics equations

In the steady-state case $(\dot{\rho}, \dot{\tilde{\rho}}, \dot{\boldsymbol{v}}, \dot{\tilde{\boldsymbol{v}}}) = 0$ the system of equations (6)–(8) has the form

$$\operatorname{div}\boldsymbol{v} = 0, \quad \operatorname{div}\tilde{\boldsymbol{v}} = 0, \tag{9}$$

$$\nu\Delta\boldsymbol{v} = \nabla p - \bar{\rho}^0\boldsymbol{f}, \tag{10}$$

$$\tilde{\nu}\Delta\tilde{\boldsymbol{v}} = \nabla p - \bar{\rho}^0\boldsymbol{f}. \tag{11}$$

Green's function $G_{ij}(\boldsymbol{r},\boldsymbol{r}')$, $\tilde{G}_{ij}(\boldsymbol{r},\boldsymbol{r}')$, $P_i(\boldsymbol{r},\boldsymbol{r}')$ $(i,j=1,2,3)$ of a steady-state system of two-velocity hydrodynamics (9)–(11) satisfies the following system of differential equations:

$$\partial_m G_{mj}(\boldsymbol{r},\boldsymbol{r}')=0, \quad \partial_m \tilde{G}_{mj}(\boldsymbol{r},\boldsymbol{r}')=0, \tag{12}$$

$$\nu\Delta G_{ij}(\boldsymbol{r},\boldsymbol{r}')-\partial_i P_j(\boldsymbol{r},\boldsymbol{r}')=\delta_{ij}\delta(\boldsymbol{r}-\boldsymbol{r}'), \tag{13}$$

$$\tilde{\nu}\Delta \tilde{G}_{ij}(\boldsymbol{r},\boldsymbol{r}')-\partial_i P_j(\boldsymbol{r},\boldsymbol{r}')=\delta_{ij}\delta(\boldsymbol{r}-\boldsymbol{r}'), \tag{14}$$

where δ_{ij} is the Kronecker delta, $\delta(\boldsymbol{r})$ is the Dirac function, ∂_i denotes the ith partial derivative, i.e., $\partial_i=\dfrac{\partial}{\partial x_i}$.

Denote by $(\hat{\boldsymbol{v}}(\boldsymbol{\alpha}),\hat{\tilde{\boldsymbol{v}}}(\boldsymbol{\alpha}),\hat{p}(\boldsymbol{\alpha}))$ the Fourier transform of $(\boldsymbol{v}(\boldsymbol{r}),\tilde{\boldsymbol{v}}(\boldsymbol{r}),p(\boldsymbol{r}))$, and, namely,

$$(\hat{\boldsymbol{v}}(\boldsymbol{\alpha}),\hat{\tilde{\boldsymbol{v}}}(\boldsymbol{\alpha}),\hat{p}(\boldsymbol{\alpha}))=\frac{1}{(2\pi)^{3/2}}\int_{R^3}(\boldsymbol{v}(\boldsymbol{r}),\tilde{\boldsymbol{v}}(\boldsymbol{r}),p(\boldsymbol{r}))\,e^{-i\boldsymbol{\alpha}\boldsymbol{r}}\,d\boldsymbol{r}.$$

Multiplying (12)–(14) by $\frac{1}{(2\pi)^{3/2}}e^{-i\boldsymbol{\alpha}(\boldsymbol{r}-\boldsymbol{r}')}$ and integrating over $\boldsymbol{r}\in R^3$, we obtain

$$\partial_m \hat{G}_{mj}=0, \quad \partial_m \hat{\tilde{G}}_{mj}=0, \qquad j=1,2,3, \tag{15}$$

$$\nu\Delta\hat{G}_{ij}-\partial_i\hat{P}_j=\frac{1}{(2\pi)^{3/2}}\delta_{ij}, \qquad i,j=1,2,3, \tag{16}$$

$$\tilde{\nu}\Delta\hat{\tilde{G}}_{ij}-\partial_i\hat{P}_j=\frac{1}{(2\pi)^{3/2}}\delta_{ij}, \qquad i,j=1,2,3. \tag{17}$$

Hence, the functions $\hat{G}_{ij}$, $\hat{\tilde{G}}_{ij}$, $\hat{P}_j$, are uniquely defined:

$$\hat{G}_{ij}=\frac{1}{(2\pi)^{3/2}\nu\alpha^2}\left[-\delta_{ij}+\frac{\alpha_i\alpha_j}{\alpha^2}\right], \qquad \hat{\tilde{G}}_{ij}=\frac{1}{(2\pi)^{3/2}\tilde{\nu}\alpha^2}\left[-\delta_{ij}+\frac{\alpha_i\alpha_j}{\alpha^2}\right],$$
$$\hat{P}_j=\frac{i\alpha_j}{(2\pi)^{3/2}\alpha^2}.$$

The inverse Fourier transform and the formulas [7]

$$\left(\delta(\boldsymbol{r}-\boldsymbol{r}'),\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|},\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{8\pi}\right)=\frac{1}{(2\pi)^3}\int_{R^3}(1,\alpha^{-2},\alpha^{-4})e^{i\boldsymbol{\alpha}(\boldsymbol{r}-\boldsymbol{r}')}d\boldsymbol{\alpha}$$

give

$$G_{kj}(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{\nu}\left[-\frac{\delta_{kj}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}+\partial_k\partial_j\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{8\pi}\right],$$
$$\tilde{G}_{kj}(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{\tilde{\nu}}\left[-\frac{\delta_{kj}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}+\partial_k\partial_j\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{8\pi}\right],$$
$$P_k(\boldsymbol{r},\boldsymbol{r}')=\partial_k\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}.$$

From these expressions we obtain Green's function of problem (9)–(11) as

$$G_{kj}(\boldsymbol{r},\boldsymbol{r}') = -\frac{1}{8\pi\nu}\left[\frac{\delta_{kj}}{|\boldsymbol{r}-\boldsymbol{r}'|} + \frac{(x_k - x'_k)(x_j - x'_j)}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\right], \tag{18}$$

$$\tilde{G}_{kj}(\boldsymbol{r},\boldsymbol{r}') = -\frac{1}{8\pi\tilde{\nu}}\left[\frac{\delta_{kj}}{|\boldsymbol{r}-\boldsymbol{r}'|} + \frac{(x_k - x'_k)(x_j - x'_j)}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\right], \tag{19}$$

$$P_k(\boldsymbol{r},\boldsymbol{r}') = -\frac{x_k - x'_k}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|^2}. \tag{20}$$

From these formulas and equations (12)–(14) it is evident that in the argument $\boldsymbol{r}'$, the functions $G_{kj}(\boldsymbol{r},\boldsymbol{r}')$, $\tilde{G}_{kj}(\boldsymbol{r},\boldsymbol{r}')$, $P_k(\boldsymbol{r},\boldsymbol{r}')$ satisfy the conjugate system

$$\frac{\partial G_{mj}(\boldsymbol{r},\boldsymbol{r}')}{\partial x'_m} = 0, \qquad \frac{\partial \tilde{G}_{mj}(\boldsymbol{r},\boldsymbol{r}')}{\partial x'_m} = 0, \tag{21}$$

$$\nu\Delta_{\boldsymbol{r}'} G_{ij}(\boldsymbol{r},\boldsymbol{r}') + \frac{\partial P_j(\boldsymbol{r},\boldsymbol{r}')}{\partial x'_i} = \delta_{ij}\delta(\boldsymbol{r}-\boldsymbol{r}'), \tag{22}$$

$$\tilde{\nu}\Delta_{\boldsymbol{r}'} \tilde{G}_{ij}(\boldsymbol{r},\boldsymbol{r}') + \frac{\partial P_j(\boldsymbol{r},\boldsymbol{r}')}{\partial x'_i} = \delta_{ij}\delta(\boldsymbol{r}-\boldsymbol{r}'), \tag{23}$$

The functions $G_{kj}(\boldsymbol{r},\boldsymbol{r}')$, $\tilde{G}_{kj}\boldsymbol{r},\boldsymbol{r}')$, $P_k(\boldsymbol{r},\boldsymbol{r}')$ allow us to construct the volume potentials

$$v_i(\boldsymbol{r}) = -\bar{\rho}^0 \int G_{ij}(\boldsymbol{r},\boldsymbol{r}') f_j(\boldsymbol{r}')\,dr', \quad \tilde{v}_i(\boldsymbol{r}) = -\bar{\rho}^0 \int \tilde{G}_{ij}(\boldsymbol{r},\boldsymbol{r}') f_j(\boldsymbol{r}')\,dr',$$

$$p(\boldsymbol{r},\omega) = \bar{\rho}^0 \int P_i(\boldsymbol{r},\boldsymbol{r}') f_i(\boldsymbol{r}')\,dr',$$

which by virtue of (12)–(14) satisfy the stationary inhomogeneous system of equations of the two-velocity hydrodynamics with one pressure (9)–(11).

The character of singularities of the kernels $G_{kj}(\boldsymbol{r},\boldsymbol{r}')$, $\tilde{G}_{kj}(\boldsymbol{r},\boldsymbol{r}')$ and $P_k(\boldsymbol{r},\boldsymbol{r}')$ is the same as for the singular solution $\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}$ to the Laplace equation and for its first derivative, respectively. This shows that the solutions of the system of equations (9)–(11) are functions of the physical densities, as well as of the bulk saturation substances constituting the two-phase continuum.

Formulas (18)–(20) for $G_{kj}(\boldsymbol{r},\boldsymbol{r}')$, $\tilde{G}_{kj}(\boldsymbol{r},\boldsymbol{r}')$ and $P_k(\boldsymbol{r},\boldsymbol{r}')$ can be obtained in another way using the vector analysis formulas

$$\operatorname{rot}\operatorname{rot}\boldsymbol{F} = -\Delta\boldsymbol{F} + \nabla \operatorname{div}\boldsymbol{F}.$$

Let us seek for $\boldsymbol{G}_k(\boldsymbol{r},\boldsymbol{r}') = (G_{k1}(\boldsymbol{r},\boldsymbol{r}'), G_{k2}(\boldsymbol{r},\boldsymbol{r}'), G_{k3}(\boldsymbol{r},\boldsymbol{r}'))$, $\tilde{\boldsymbol{G}}_k(\boldsymbol{r},\boldsymbol{r}') = (\tilde{G}_{k1}(\boldsymbol{r},\boldsymbol{r}'), \tilde{G}_{k2}(\boldsymbol{r},\boldsymbol{r}'), \tilde{G}_{k3}(\boldsymbol{r},\boldsymbol{r}'))$ in the form

$$\boldsymbol{G}_k = \operatorname{rot}\operatorname{rot}\boldsymbol{U}_k = -\Delta\boldsymbol{U}_k + \nabla\operatorname{div}\boldsymbol{U}_k,$$
$$\tilde{\boldsymbol{G}}_k = \operatorname{rot}\operatorname{rot}\tilde{\boldsymbol{U}}_k = -\Delta\tilde{\boldsymbol{U}}_k + \nabla\operatorname{div}\tilde{\boldsymbol{U}}_k.$$

Substituting these relations into (13), (14) and separating the potential parts from the solenoidal ones, we obtain

$$-\nu\Delta^2\boldsymbol{U}_k = \delta(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{e}^k, \tag{24}$$
$$-\tilde{\nu}\Delta^2\tilde{\boldsymbol{U}}_k = \delta(\boldsymbol{r}-\boldsymbol{r}')\boldsymbol{e}^k, \tag{25}$$
$$P_k = \nu\operatorname{div}\Delta\tilde{\boldsymbol{U}}_k = \tilde{\nu}\operatorname{div}\Delta\tilde{\boldsymbol{U}}_k. \tag{26}$$

Hence,

$$\boldsymbol{U}_k(\boldsymbol{r},\boldsymbol{r}') = \frac{\boldsymbol{e}^k}{8\pi\nu}|\boldsymbol{r}-\boldsymbol{r}'|, \quad \tilde{\boldsymbol{U}}_k(\boldsymbol{r},\boldsymbol{r}') = \frac{\boldsymbol{e}^k}{8\pi\tilde{\nu}}|\boldsymbol{r}-\boldsymbol{r}'|,$$
$$P_k(\boldsymbol{r},\boldsymbol{r}') = \operatorname{div}\frac{\boldsymbol{e}^k}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{\partial}{\partial x_k}\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}.$$

These formulas, as is easily seen, coincide with formulas (18)–(20). In formulas (24)–(26), $\boldsymbol{e}^k$ is the unit vector along the kth coordinate axis.

5. Potentials of single and double layers

As in the one-velocity hydrodynamics, we introduce the stress tensor in the case of the two-velocity hydrodynamics:

$$T_{ik}(\boldsymbol{v}) = -p\delta_{ik} + \nu(\partial_i v_k + \partial_k v_i),$$

and

$$\tilde{T}_{ik}(\tilde{\boldsymbol{v}}) = -p\delta_{ik} + \tilde{\nu}(\partial_i \tilde{v}_k + \partial_k \tilde{v}_i),$$

corresponding to $(\boldsymbol{v}, p)$ and $(\tilde{\boldsymbol{v}}, p)$.

With the arguments from [7], an analog of Green's formula corresponding to the Stokes problem for the two-velocity hydrodynamics is obtained:

$$\int_\Omega (\nu\Delta v_i - \partial_i p)u_i\,d\boldsymbol{r} = -\int_\Omega \frac{\nu}{2}(\partial_i u_k + \partial_k u_i)(\partial_i v_k + \partial_k v_i)\,d\boldsymbol{r} + \int_S T_{ik}(\boldsymbol{v})u_i n_k\,dS, \tag{27}$$

$$\int_\Omega (\tilde{\nu}\Delta \tilde{v}_i - \partial_i p)\tilde{u}_i\,d\boldsymbol{r} = -\int_\Omega \frac{\tilde{\nu}}{2}(\partial_i \tilde{u}_k + \partial_k \tilde{u}_i)(\partial_i \tilde{v}_k + \partial_k \tilde{v}_i)\,d\boldsymbol{r} + \int_S \tilde{T}_{ik}(\tilde{\boldsymbol{v}})u_i n_k\,dS, \tag{28}$$

where $\boldsymbol{n} = (n_1, n_2, n_3)$ is the external to the domain $\Omega \subset R^3$ normal to S. Interchanging $\boldsymbol{u}$, $\tilde{\boldsymbol{u}}$ and $\boldsymbol{v}$, $\tilde{\boldsymbol{v}}$ and introducing along with p any smooth function q, from (27) and (28) we obtain the following formulas:

$$\int_\Omega [(\nu\Delta v_i - \partial_i p)u_i - v_i(\nu\Delta u_i - \partial_i q)]\,d\boldsymbol{r}$$
$$= \int_S [T_{ik}(\boldsymbol{v})u_i n_k - T'_{ik}(\boldsymbol{u})v_i n_k]\,dS, \tag{29}$$
$$\int_\Omega [(\tilde{\nu}\Delta\tilde{v}_i - \partial_i p)\tilde{u}_i - \tilde{v}_i(\nu\Delta\tilde{u}_i - \partial_i q)]\,d\boldsymbol{r}$$
$$= \int_S [\tilde{T}_{ik}(\tilde{\boldsymbol{v}})\tilde{u}_i n_k - \tilde{T}'_{ik}(\tilde{\boldsymbol{u}})\tilde{v}_i n_k]\,dS, \tag{30}$$

where

$$T'_{ik}(\boldsymbol{u}) = q\delta_{ik} + \nu(\partial_i u_k + \partial_k u_i), \quad \tilde{T}'_{ik}(\tilde{\boldsymbol{u}}) = q\delta_{ik} + \tilde{\nu}(\partial_i\tilde{u}_k + \partial_k\tilde{u}_i).$$

Equalities (27)–(30) are called Green's formulas corresponding to the Stokes problem for the two-velocity hydrodynamics. Using (29), (30) and the usual method of singular solutions, a representation of any solution $\boldsymbol{v}$, $\tilde{\boldsymbol{v}}$, p of the inhomogeneous system (9)–(11) through the term $\boldsymbol{f}$ and the values of $\boldsymbol{v}$, $\tilde{\boldsymbol{v}}$ and $T_{ik}(\boldsymbol{v})$, $\tilde{T}_{ik}(\tilde{\boldsymbol{v}})$ on S, is obtained.

Potentials of a simple layer with the density $\phi(\xi) = (\phi_1(\xi), \phi_2(\xi), \phi_3(\xi))$ are of the form [7]:

$$\boldsymbol{V}(\boldsymbol{r},\boldsymbol{\phi}) = -\int_S \boldsymbol{G}_k(\boldsymbol{r},\xi)\phi_k(\xi)\,dS_\xi, \quad Q(\boldsymbol{r},\boldsymbol{\phi}) = -\int_S P_k(\boldsymbol{r},\xi)\phi_k(\xi)\,dS_\xi,$$
$$\tilde{\boldsymbol{V}}(\boldsymbol{r},\boldsymbol{\phi}) = -\int_S \tilde{\boldsymbol{G}}_k(\boldsymbol{r},\xi)\phi_k(\xi)\,dS_\xi,$$

and the double-layer potentials with the density $\psi(\xi) = (\psi_1(\xi), \psi_2(\xi), \psi_3(\xi))$ are called integrals of the form

$$W_k(\boldsymbol{r},\boldsymbol{\psi}) = \int_S T'_{ij}(\boldsymbol{G}_k(\boldsymbol{r},\xi))\psi_j(\xi)n_j(\xi)\,dS_\xi,$$
$$\tilde{W}_k(\boldsymbol{r},\boldsymbol{\psi}) = \int_S \tilde{T}'_{ij}(\tilde{\boldsymbol{G}}_k(\boldsymbol{r},\xi))\psi_j(\xi)n_j(\xi)\,dS_\xi,$$
$$\Pi(\boldsymbol{r},\boldsymbol{\psi}) = -2\nu\frac{\partial}{\partial x_j}\int_S P_k(\boldsymbol{r},\xi)n_j(\xi)\psi_k(\xi)\,dS_\xi,$$
$$\tilde{\Pi}(\boldsymbol{r},\boldsymbol{\psi}) = -2\tilde{\nu}\frac{\partial}{\partial x_j}\int_S P_k(\boldsymbol{r},\xi)n_j(\xi)\psi_k(\xi)\,dS_\xi.$$

All considerations up to now have belonged to the three-dimensional case. Let us present the results for the two-dimensional case. A singular solution of the two-velocity hydrodynamics equation (9)–(11) has the form

$$
\begin{aligned}
G_{kj}(\boldsymbol{r}, \boldsymbol{r}') &= -\frac{1}{4\pi\nu}\left[\delta_{kj} \ln \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} + \frac{(x_k - x'_k)(x_j - x'_j)}{|\boldsymbol{r} - \boldsymbol{r}'|^2}\right], \\
\tilde{G}_{kj}(\boldsymbol{r}, \boldsymbol{r}') &= -\frac{1}{4\pi\tilde{\nu}}\left[\delta_{kj} \ln \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} + \frac{(x_k - x'_k)(x_j - x'_j)}{|\boldsymbol{r} - \boldsymbol{r}'|^2}\right], \\
P_k(\boldsymbol{r}, \boldsymbol{r}') &= \frac{1}{2\pi} \frac{\partial}{\partial x_k} \ln \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}.
\end{aligned}
$$

As for the electrostatic potentials, there is one significant difference between the two-dimensional and the three-dimensional cases due to the fact that $G_{kj}, \tilde{G}_{kj}$ with $|\boldsymbol{r}| \to \infty$ behaves as $\ln \frac{1}{|\boldsymbol{r}|}$ in the first case and as $\frac{1}{|\boldsymbol{r}|}$ in the second one. It can be shown that in the case of the analog to the Stokes problem for the stationary equation of the two-velocity hydrodynamics in the two-dimensional version, the first boundary value problem for a system of biharmonic equations for the current functions is obtained.

6. Conclusion

To describe the three-dimensional steady-state viscous fluids of the two-velocity continuum with pressure phase equilibrium, the fundamental solution has been obtained. The effect of physical phase densities, saturation volume substances and viscosity of the two-phase continuum flows on the velocity and pressure of currents is shown.

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