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Regularization of an inverse dynamic problem for the equation of SH waves in a porous medium*

Kh.Kh. Imomnazarov, Sh.Kh. Imomnazarov, T.T. Rakhmonov, Z.Sh. Yangiboev

Abstract. A regularizing algorithm of an inverse problem for a one-dimensional equation of SH waves in a fluid-saturated porous medium with energy loss under intercomponent friction is proposed.

1. Introduction

Many engineering problems are reduced to solving purely mathematical problems. Going from engineering problems to purely mathematical ones can pose certain complications. Therefore, the creation of mathematical models for physical processes is among the most important areas in modern science. Boundary value problems have found a wide use in the fluid mechanics. These are problems, in which either the shape of an object (the underground contour of a dam, oil-water contact, airfoil contour, etc.) is found using given characteristics or the characteristics are calculated with its shape given. The former problems are called the direct boundary value problems, and the latter – the inverse boundary value problems [1]. Specifically, such problems emerge in exploration geophysics when searching for oil strata and choosing parameters of the wave action on oil and gas deposits to intensify oil production. The development of models for filtration in porous media, which is determining in solving geophysical problems, started in the second half of the 19th century. The development of many problems of filtration was based on the law of resistance in fluid filtration, experimentally established in 1856 by H. Darcy, a French engineer. The law reveals that the fluid filtration rate is proportional to the pressure gradient. The filtration coefficient characterizes both a medium and a fluid, that is, it depends on the size of particles, their shape and roughness, the porosity and permeability of the medium, and the viscosity of the fluid. The first theoretical investigations into filtration based on this law were made by J. Dupuit and then continued by F. Forchheimer. The first two-velocity mathematical models for the description of seismic wave propagation in fluid-saturated porous media were developed in [2, 3]. A non-isothermal model of filtration under

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assumption of the entropy additivity of porous medium components was obtained with the use of conservation laws by P.H. Roberts and D.E. Loper [4]. A continual theory of filtration not limited by such an assumption was constructed in [5, 6], also, within the method of conservation laws. The Darcy law is obtained as a consequence of the equations of the above-mentioned theories in one of the limiting cases.

In recent decades, much attention of mathematicians has been given to the so-called ill-posed problems, that is, problems whose solution may not exist or be non-unique, unstable. Among such problems are many inverse initial boundary value problems of mathematical physics. Some mathematical statements of inverse problems in wave propagation theory for a model of elastic media were first considered by A.S. Alekseev [7, 8]. It was found that they are linked with one-dimensional inverse spectral problems considered by I.M. Gelfand and B.M. Levitan [9] and M.G. Krein [10, 11]. In [12], a relation between the Baranov–Kunetz method and a discrete analog of the Gel'fand–Levitan method was established. In this case, the conditions of solvability of the Gel'fand–Levitan equations made possible to correct inaccurate seismograms. A rather extensive bibliography on the theory of inverse problems for hyperbolic-type equations can be found in [13–23].

In this paper, using the idea from [23], we construct a regularizing algorithm of an inverse problem for a one-dimensional equation of SH waves in fluid-saturated porous media with energy loss under intercomponent friction.

2. Problem statement

Let a half-space $z > 0$ be filled with a non-homogeneous porous medium. Equations for the propagation of seismic SH waves with allowance for energy absorption caused by the intercomponent friction coefficient $b(z)$ have the following form [24, 25]:

$$\rho_s(z)\, u_{tt} = (\mu(z)\, u_z)_z - \rho_l(z)b(z)\,(u_t - v_t), \tag{1}$$

$$\rho_l(z)\, v_{tt} = \rho_l(z)b(z)\,(u_t - v_t). \tag{2}$$

Here u and v are the velocity vector components of the displacement of particles of an elastic porous body and fluid with the partial densities $\rho_s(z)$ and $\rho_l(z)$, respectively. Assume the porous medium be at rest at $t < 0$:

$$u|_{t=0} = u_t|_{t=0} = 0, \qquad v|_{t=0} = v_t|_{t=0} = 0. \tag{3}$$

Let the following force be applied at the boundary $z = 0$:

$$\mu u_z|_{z=0} = \delta(t). \tag{4}$$

Here $\delta(t)$ is the Dirac delta function.

Using the information from (4), once continuously differentiable positive functions $\rho_s(z)$, $\mu(z)$, and continuous positive functions $\rho_l(z)$, $b(z)$, it is necessary to determine twice continuously differentiable functions $u(t,z)$, $v(t,z)$ from (1)–(3). This problem will be called a direct dynamic problem for equations of SH waves in a porous medium.

In applications, of great interest are problems of determining the variable coefficients of a differential equation. This is due to the fact that, as a rule, differential equations describe physical processes, and the equation coefficients are associated with the physical characteristics of the medium in which these processes occur. Since these coefficients cannot be measured directly, the problem of determining the properties of a substance is, in essence, an inverse one.

Using the method proposed in [23] for the inverse problem of elasticity theory, we construct a regularizing algorithm of the following inverse problem:

Problem 1. Using the information $u|_{z=0} = \phi(t)$, reconstruct $\mu(z)$ from (1)–(4). The other functions, $\rho_s(z)$, $\rho_l(z)$, and $b(z)$, are considered to be known.

3. Reducing problem (1)–(4) to a canonical form

Introduce, instead of z, the coordinate

$$x = \int_0^z \frac{d\xi}{c_t(\xi)},$$

where $c_t(z) = \sqrt{\dfrac{\mu(z)}{\rho_s(z)}}$ is the propagation velocity of the transverse seismic waves in a porous medium.

With the use of the coordinate x, the propagation velocity of seismic waves in a porous medium becomes unity. Since $\dfrac{\partial}{\partial z} = \dfrac{1}{c_t}\dfrac{\partial}{\partial x}$, equations (1) and (2) have the canonical form

$$u_{tt} - u_{xx} = (\ln\sigma)' u_x - b(x)\frac{\rho_l(x)}{\rho_s(x)}(u_t - v_t), \quad x > 0, \tag{5}$$

$$v_t = b(x)\,(u - v), \quad x > 0, \tag{6}$$

$$u|_{t=0} = u_t|_{t=0} = 0, \quad v|_{t=0} = 0, \tag{7}$$

$$u_x|_{x=0} = \frac{\delta(t)}{\sigma(0)}. \tag{8}$$

In formula (5), $\sigma(x) = \sqrt{\mu(x)\rho_s(x)}$ is the acoustic stiffness, $\sigma > 0$. Now assume that inequalities

$$\begin{aligned} 0 < \rho_s^{\min} \le \rho_s(x) \le \rho_s^{\max} < \infty, \\ 0 < \rho_l^{\min} \le \rho_l(x) \le \rho_l^{\max} < \infty, \\ 0 < b^{\min} \le b(x) \le b^{\max} < \infty \end{aligned} \tag{9}$$

are satisfied.

Inverse Problem 1 is reformulated as follows: let, on the interval $[0, T]$, the function

$$u|_{x=0} = \phi(t), \qquad t \in [0, T], \tag{10}$$

be specified, and it is necessary to determine $\sigma(x)$, $x \in [0, T/2]$. Let A denote the operator for solving the direct problem, $\phi = A \ln \sigma$, $\sigma \in C^1[0, T/2]$, and let Φ be the image of the space $C^1[0, T/2]$ at the mapping A.

According to [7, 8, 21–23], one can show that Φ is a set in $C^1[0, T]$ determined by the equality

$$\Phi = \Big\{ \phi \in C^1[0, T] : \ \phi(0) < 0,$$
$$\|\psi\|^2_{L_2(0,T)} + \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \phi'(|t-s|)\psi(t)\psi(s)\, dt\, ds \ge 0, \ \ \forall \psi \in L_2(0, T) \Big\},$$

and the operator A is the homeomorphism of $C^1[0, T]$ on Φ.

4. The regularizing algorithm of problem (5)–(8)

Let, instead of the function $\phi = A \ln \sigma$, its approximate value $\tilde{\phi} \in C^1[0, T]$, $\|\phi - \tilde{\phi}\| \le \delta$ be known. If $\tilde{\phi} \in \Phi$, due to the continuity of A^{-1} on Φ, as an approximate value of σ one can take $\tilde{\sigma} = \exp[A^{-1}\tilde{\phi}]$. Assume that $\tilde{\phi} \in C^1[0, T]$, $\tilde{\phi}(0) < 0$, but, generally speaking, it does not belong to Φ. In this case, a natural method of finding an approximation is to construct the mapping $R : \tilde{\Phi} \to C^1[0, T/2]$, $\tilde{\Phi} = \{\phi \in C^1[0, T] : \ \phi(0) < 0\}$ approximating the inverse operator A^{-1} on Φ and determined on the entire set $\tilde{\Phi}$ [7, 8, 21–23].

As shown in [21, 22], the initial inverse problem is reduced to a Volterra-type nonlinear equation. In this paper, according to [23], a regularizing algorithm preserving its “Volterra properties” is proposed.

First of all, let us show that the operator A acts from $C^1[0, T/2]$ to $C^1[0, T]$, and $(A \ln \sigma)(0) = -1/\sigma(0) < 0$. Let $\sigma \in C^1[0, T/2]$. A solution to problem (5)–(8) is sought for in the class of piecewise smooth functions of the form

$$u(x, t) = \theta(t - x)\, u_\Delta(x, t), \quad v(x, t) = \theta(t - x)\, v_\Delta(x, t).$$

Here $\theta(t)$ is the Heaviside function, and u_Δ, v_Δ are the restrictions of u, v to the closed domain $\Delta = \{(x, t) : 0 \le x \le t\}$, $u_\Delta, v_\Delta \in C^1(\Delta)$.

Then it follows from (5)–(8) that at any $T > 0$ the restrictions of u, v to the triangle $\Delta(T) = \{(x,t) : 0 \le x \le t \le T - x\}$ (again denoted by u, v) must satisfy the following relations:

$$u_{tt} - u_{xx} = (\ln\sigma)' u_x - b(x)\frac{\rho_l(x)}{\rho_s(x)}(u_t - v_t), \quad 0 < x < t < T - x, \tag{11}$$

$$v_t = b(x)\,(u - v), \quad 0 < x < t < T - x, \tag{12}$$

$$u_x|_{x=0} = 0, \quad 0 \le t \le T, \tag{13}$$

$$u(x,x) = -\frac{1}{\sqrt{\sigma(0)\sigma(x)}}\exp\Big(-\int_0^x \frac{b(y)\rho_l(y)}{2\rho_s(y)}\,dy\Big), \quad 0 \le x \le T/2, \tag{14}$$

$$v|_{t=0} = 0, \quad 0 \le x \le T/2, \tag{15}$$

It is easy to see that problem (11)–(15) is equivalent to the equations

$$\begin{aligned} u(x,t) = {} & \omega\Big(\frac{t+x}{2}\Big) + \omega\Big(\frac{t-x}{2}\Big) - \omega(0) - \\ & \frac{1}{2}\int_0^x (\ln\sigma)'(\xi)\,d\xi \int_{t-x+\xi}^{t+x-\xi} U(\xi,\zeta)\,d\zeta + \\ & \frac{1}{2}\int_0^{(t+x)/2} (\ln\sigma)'(\xi)\,d\xi \int_{\xi}^{t+x-\xi} U(\xi,\zeta)\,d\zeta + \\ & \frac{1}{2}\int_0^{(t-x)/2} (\ln\sigma)'(\xi)\,d\xi \int_{\xi}^{t-x-\xi} U(\xi,\zeta)\,d\zeta + \\ & \frac{1}{2}\int_0^x b(\xi)\frac{\rho_l(\xi)}{\rho_s(\xi)}\,d\xi \int_{t-x+\xi}^{t+x-\xi} V(\xi,\zeta)\,d\zeta - \\ & \frac{1}{2}\int_0^{(t+x)/2} b(\xi)\frac{\rho_l(\xi)}{\rho_s(\xi)}\,d\xi \int_{\xi}^{t+x-\xi} V(\xi,\zeta)\,d\zeta - \\ & \frac{1}{2}\int_0^{(t-x)/2} b(\xi)\frac{\rho_l(\xi)}{\rho_s(\xi)}\,d\xi \int_{\xi}^{t-x-\xi} V(\xi,\zeta)\,d\zeta, \end{aligned} \tag{16}$$

$$v(x,t) = b(x)\int_0^t \mathrm{e}^{-b(x)(t-s)} u(x,s)\,ds, \tag{17}$$

where $\omega(x) = u(x,x)$, $U = u_x$, $W = u_t$,

$$V(\xi,\zeta) = W(\xi,\zeta) - b(\xi)\int_0^{\zeta} \mathrm{e}^{-b(\xi)(\zeta-s)} u(\xi,s)\,ds.$$

Differentiating (16) with respect to x and t, we obtain integral equations for the functions $U(x,t)$, $W(x,t)$, and $u(x,t)$, whose solution exists and is unique in $C(\Delta(T))$. Substituting $U(x,t)$ and $W(x,t)$ into (16), we find the solution $u(x,t)$ from the class C^1 to problem (11), (13), (14). Substituting

$u(x,t)$ into (17), we find the solution $v(x,t)$ of the Cauchy problem (12), (15). Hence, the function $\phi(t) = u|_{x=0}$ will be from $C^1[0,T]$ and $\phi(0) = -1/\sigma(0) < 0$, that is, $\phi \in \tilde{\Phi}$.

Let us show that the equation $A \ln \sigma = \phi$, $\phi \in \Phi$, is equivalent to the Volterra equation. For this, we consider the Banach space Z of the vector-functions

$$z(x,t) = (z_1(x,t), z_2(x,t), z_3(x), z_4(x)),$$

which are continuous on $\Delta(T)$, with a naturally defined operation of multiplication by the scalar functions from $C(\Delta(T))$ and the norm

$$\|z\| = \max\{\|z_1\|, \|z_2\|, \|z_3\|, \|z_4\|\}.$$

In Z, we consider a subset $\tilde{Z}_0$ consisting of vector-functions $z_0 \equiv P\phi$, $\phi \in \tilde{\Phi}$, of the form

$$z_0(x,t) = \left\{\frac{\phi'(t+x) - \phi'(t-x)}{2}, \frac{\phi'(t+x) + \phi'(t-x)}{2}, \phi'(2x), \frac{1}{\phi(0)}\right\}. \quad (18)$$

It is evident that there is a one-to-one correspondence between $\tilde{Z}_0$ and $\tilde{\Phi}$. If in (20) $\phi \in \tilde{\Phi}$, we have $z_0 \in \tilde{Z}_0$. Let us determine the operator $M : Z \times \bar{R}_+ \to Z$, $\bar{R}_+ = \{t : t \geq 0\}$, by the formulas

$$\begin{aligned}
(M(z,\alpha))_1(x,t) = &\int_0^x f(z,\alpha)(\xi)[z_1(\xi, t+x-\xi) + z_1(\xi, t-x+\xi)]\, d\xi + \\
&\frac{1}{2}\int_0^x b(\xi)\frac{\rho_l(\xi)}{\rho_s(\xi)}\Big\{ z_1(\xi, t+x-\xi) + z_1(\xi, t-x+\xi) + \\
&\quad z_2(\xi, t+x-\xi) + z_2(\xi, t-x+\xi) - \\
&\quad b(\xi)\int_0^{t+x-\xi} \mathrm{e}^{-b(\xi)(t+x-\xi-s)} z_2(\xi, s)\, ds - \\
&\quad b(\xi)\int_0^{t-x+\xi} \mathrm{e}^{-b(\xi)(t-x+\xi-s)} z_2(\xi, s)\, ds \Big\}\, d\xi,
\end{aligned}$$

$$\begin{aligned}
(M(z,\alpha))_2(x,t) = &\int_0^x f(z,\alpha)(\xi)[z_1(\xi, t+x-\xi) - z_1(\xi, t-x+\xi)]\, d\xi + \\
&\frac{1}{2}\int_0^x b(\xi)\frac{\rho_l(\xi)}{\rho_s(\xi)}\Big\{ z_1(\xi, t+x-\xi) - z_1(\xi, t-x+\xi) + \\
&\quad z_2(\xi, t+x-\xi) - z_2(\xi, t-x+\xi) - \\
&\quad b(\xi)\int_0^{t+x-\xi} \mathrm{e}^{-b(\xi)(t+x-\xi-s)} z_2(\xi, s)\, ds + \\
&\quad b(\xi)\int_0^{t-x+\xi} \mathrm{e}^{-b(\xi)(t-x+\xi-s)} z_2(\xi, s)\, ds \Big\}\, d\xi,
\end{aligned}$$

$$
\begin{aligned}
(M(z,\alpha))_3(x) = 2\int_0^x & f(z,\alpha)(\xi)\, z_1(\xi, 2x-\xi)\, d\xi \, + \\
& \int_0^x b(\xi)\frac{\rho_l(\xi)}{\rho_s(\xi)}\Big\{ z_1(\xi, 2x-\xi) + z_2(\xi, 2x-\xi) - \\
& \qquad b(\xi)\int_0^{2x-\xi} \mathrm{e}^{-b(\xi)(2x-\xi-s)} z_2(\xi, s)\, ds \Big\}\, d\xi, \\
(M(z,\alpha))_4(x) = -\int_0^x & f(z,\alpha)(\xi) z_4(\xi)\, d\xi, \qquad (19)
\end{aligned}
$$

where $f(z,\alpha) = \frac{z_3 z_4}{(1+\alpha z_3^2)(1+\alpha z_4^2)}$.

Lemma 1. *The equation $z = z_0 + M(z,0)$, $z_0 \in \tilde{Z}_0$, is solvable in Z if and only if $z_0 \in Z_0$.*

Proof. Let $z_0 \in \tilde{Z}_0$. By the definition of the sets Z_0 and Φ, this means that there exists a function $\sigma \in C^1[0, T/2]$, $\sigma > 0$, such that $A \ln \sigma = \phi$, where ϕ is uniquely determined by the function z_0 according to (18). Then, by definition, $\phi(t) = u|_{x=0}$, $t \in [0,T]$, where $u(x,t)$ is the solution to problem (11), (13), (14) with the function $\sigma = \exp[A^{-1}\phi]$. Let us show that in this case the vector-function

$$
z(x,t) = (u_x(x,t), u_t(x,t), [u(x,x)]', 1/u(x,x))
$$

satisfies the equation $z = z_0 + M(z,0)$. In fact, by inverting the wave operator $\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}$ by the D'Alembert formula with allowance for the Cauchy data $u|_{x=0} = \phi(t)$, $u_x|_{x=0} = 0$ and the relation following from (14),

$$
f(z,0) + \frac{b\rho_l}{2\rho_s} = \frac{u'}{u} + \frac{b\rho_l}{2\rho_s} = -\frac{1}{2}(\ln\sigma)',
$$

we find

$$
\begin{aligned}
u(x,t) = \frac{\phi(t+x)+\phi(t-x)}{2} + \int_0^x f(z,0)(\xi)\, d\xi \int_{t-x+\xi}^{t+x-\xi} z_1(\xi,\zeta)\, d\zeta \, + \\
\frac{1}{2}\int_0^x b(\xi)\frac{\rho_l(\xi)}{\rho_s(\xi)}\, d\xi \int_{t-x+\xi}^{t+x-\xi} \Big\{ z_1(\xi,\zeta) + z_2(\xi,\zeta) - \\
b(\xi)\int_0^{\zeta} \mathrm{e}^{-b(\xi)(\zeta-s)} z_2(\xi,s)\, ds \Big\}\, d\zeta. \qquad (20)
\end{aligned}
$$

Hence, differentiation with respect to x results in the equality

$$
z_1(x,t) = z_{01}(x,t) + (M(z,0))_1(x,t).
$$

Differentiating the both parts of equality (20) with respect to t, we obtain

$$z_2(x,t) = z_{02}(x,t) + (M(z,0))_2(x,t).$$

Then, setting $t = x$ in (20), we obtain by differentiation $z_3(x) = z_{03}(x) + (M(z,0))_3(x)$. Finally, by definition $z_4'(x) = -z_3(x)z_4^2(x)$. Hence $z_4(x) = z_{04} + (M(z,0))_4(x)$.

Conversely, let $z \in \tilde{Z}$ be the solution to the equation

$$z = z_0 + M(z,0), \quad z_0 \in \tilde{Z}_0.$$

Then the function $z_4(x) \in C^1[0,T/2]$ satisfies the relation $z_4' + f(z,0)z_4 = 0$ and the condition $z_4(0) = 1/\phi(0) < 0$. Hence, it is negative everywhere. We set

$$u(x,t) = \phi(t) + \int_0^x z_1(\xi,t)\,d\xi, \quad (x,t) \in \Delta(T), \tag{21}$$

$$\sigma(x) = -z_4^2(x)\exp\Big\{-\int_0^x b(y)\frac{\rho_l(y)}{\rho_s(y)}\,dy\Big\}, \quad x \in [0,T/2],$$

where ϕ is the function from $\tilde{\Phi}$ corresponding to z_0. Let us show that the pair (u,σ) satisfies equalities (11), (13), and (14). In fact, by definition $u_x = z_1$, $\sigma'/\sigma = 2z_4'/z_4 = -2f(z,0)$. Hence

$$\begin{aligned} u(x,t) = \phi(t) + \frac{1}{2}\int_0^x \frac{\partial}{\partial\xi}\Big\{&\phi(t+\xi) + \phi(t-\xi) - \\ &\int_0^\xi \frac{\sigma'(\eta)}{\sigma(\eta)}\,d\eta \int_{t-\xi+\eta}^{t+\xi-\eta} u_\eta(\eta,\zeta)\,d\zeta + \\ &\int_0^\xi b(\eta)\,\frac{\rho_l(\eta)}{\rho_s(\eta)}\,d\eta \int_{t-\xi+\eta}^{t+\xi-\eta}\Big[u_\zeta(\eta,\zeta) - \\ &\quad b(\eta)\int_0^\zeta \mathrm{e}^{-b(\eta)(\zeta-s)}u_s(\eta,s)\,ds\Big]\,d\zeta\Big\}d\xi \\ = \frac{\phi(t+x)+\phi(t-x)}{2} &- \frac{1}{2}\int_0^x \frac{\sigma'(\xi)}{\sigma(\xi)}\,d\xi \int_{t-x+\xi}^{t+x-\xi} u_\xi(\xi,\zeta)\,d\zeta + \\ \frac{1}{2}\int_0^x b(\xi)\frac{\rho_l(\xi)}{\rho_s(\xi)}\,d\xi &\int_{t-x+\xi}^{t+x-\xi}\Big\{u_\zeta(\xi,\zeta) - \\ &b(\xi)\int_0^\zeta \mathrm{e}^{-b(\xi)(\zeta-s)}u_s(\xi,s)\,ds\Big\}d\zeta. \end{aligned}$$

One can see that $u \in C^1(\Delta(T))$ and equalities (11) and (13) are valid.

Let us verify whether (14) is satisfied. Setting $t = x$ in (21) and differentiating, we find

$$\frac{d}{dx}u(x,x) = z_3(x) = \frac{d}{dx}\Big(\frac{1}{z_4(x)}\Big).$$

Since $z_4 < 0$, from (21) we have the equality

$$z_4(x) = -\sqrt{\sigma(0)\sigma(x)}\exp\Big\{\int_0^x \frac{b(y)\rho_l(y)}{2\rho_s(y)}\,dy\Big\}.$$

Then

$$u(x,x) + \frac{1}{\sqrt{\sigma(0)\sigma(x)}}\exp\Big\{-\int_0^x \frac{b(y)\rho_l(y)}{2\rho_s(y)}\,dy\Big\} = u(x,x) - \frac{1}{z_4(x)} = 0.$$

Thus, the pair (u,σ), $u \in C^1(\Delta(T))$, $\sigma \in C^1[0,T/2]$, $\sigma > 0$, satisfies equalities (11)–(15). It follows from the definition of the set Φ that the function $\phi(t) = u|_{x=0}$ belongs to Φ. Hence, $z_0 \in \tilde{Z}_0$. □

Thus, it has been found that the solutions to the equations $A\ln\sigma = \phi$, $\phi \in \Phi$, and $z = z_0 + M(z,0)$, $z_0 \in \tilde{Z}_0$, are equivalent. Now, we specify, instead of the function $\phi \in \Phi$, its approximate value $\tilde{\phi} \in \tilde{\Phi}$, $\|\phi - \tilde{\phi}\| \le \delta$. For simplicity, we assume that $\phi(0) = \tilde{\phi}(0)$ (the inequality $\phi(0) \ne \tilde{\phi}(0)$ does not introduce any fundamental changes). In terms of the functions $z_0 = P\phi$ and $\tilde{z}_0 = P\tilde{\phi}$, this means that $z_0 \in Z_0$, $\tilde{z}_0 \in \tilde{Z}_0$ and $\|z_0 - \tilde{z}_0\| \le \delta$. If z_0 does not belong to Z_0, then according to Lemma 1 the equation $z = z_0 + M(z,0)$ does not have any solution.

Now, consider the regularized equation $z = z_0 + M(z,\alpha)$ $\alpha > 0$. Let B_r be a ball in Z of radius r, $B_r = \{z \in Z:\ \|z\| \le r\}$, and

$$\|z\|(x) = \max\Big\{\sup_{x\le t\le T-x}|z_1(x,t)|, |z_2(x,t)|, |z_3(x)|, |z_4(x)|\Big\}, \qquad z \in Z.$$

Lemma 2.

1. *$M \in C^1(Z\times\bar{R}_+; Z)$, that is, the operator M is continuous from $Z\times\bar{R}_+$ to Z and has the continuous partial derivatives $M_z(z,\alpha)$ and $M_\alpha(z,\alpha)$.*
2. *For any $z \in Z$, $\alpha > 0$*

$$\|M(z,\alpha)\|(x) \le \frac{c_1}{2\alpha}\int_0^x \|z\|(\xi)\,d\xi, \quad x \in [0,T/2], \tag{22}$$

where $c_1(\alpha,T) = \Big(1 + 2b^{\max}\frac{\rho_l^{\max}}{\rho_s^{\min}}\Big)(1 + 2Tb^{\max})$.

3. *For any $r > 0$, $\alpha > 0$, $z \in B_r$, and $y \in B_r$*

$$\|M(z,\alpha) - M(y,\alpha)\|(x) \le c_2(r,\alpha,T)\int_0^x \|z - y\|(\xi)\,d\xi, \quad x \in [0,T/2], \tag{23}$$

where $c_2(r,\alpha,T) = \Big(1 + 4r\sqrt{\alpha} + b^{\max}\frac{\rho_l^{\max}}{\rho_s^{\min}}\alpha\Big)(1 + 2Tb^{\max})/(2\alpha)$.

Proof. The first statement is evident and is verified by direct calculations. Note that the operator M_z at any fixed z, α is a linear continuous Volterra operator in $L(Z, Z)$. Let us prove inequality (22). It follows from the definition of the operator M that for any $\alpha > 0$:

$$\|M(z,\alpha)\|(x) \le 2\int_0^x \Big(|f(z,\alpha)| + \frac{1}{2}b(\xi)\frac{\rho_l(\xi)}{\rho_s(\xi)}\Big)\|z\|(\xi)\,d\xi + \frac{1}{2}\int_0^x b(\xi)\frac{\rho_l(\xi)}{\rho_s(\xi)}(1+2Tb(\xi))2\|z\|(\xi)\,d\xi.$$

Hence, with allowance for (9) and the inequality $|f(z,\alpha)| \le 1/(4\alpha)$ [23], we obtain inequality (22). Inequality (23) is similarly proved if we take into account that the function $f(z,\alpha)$ satisfies the inequality [23]:

$$|f(z,\alpha) - f(y,\alpha)| \le \frac{1}{\sqrt{\alpha}}\max\{|z_1 - y_1|, |z_2 - y_2|\}. \qquad \square$$

Consider the regularized equation

$$z = z_0 + M(z,\alpha). \tag{24}$$

Theorem 1. *Let $z_0 \in Z$. Then, for any $\alpha > 0$ in Z there exists a unique solution $z(\alpha)$ to equation (24). In addition, as a function of the parameter α, it is continuously differentiable in R_+ and*

$$\|z(\alpha)\| \le \|z_0\|\exp\Big(\frac{c_1T}{4\alpha}\Big). \tag{25}$$

Proof. First, find a priori estimate (25). Let $z(\alpha) \in Z$ be a solution corresponding to the value $\alpha > 0$. It follows from inequality (22) that

$$\|z(\alpha)\|(x) \le \|z_0\| + \frac{c_1}{2\alpha}\int_0^x \|z(\alpha)\|(\xi)\,d\xi, \quad x \in [0, T/2], \tag{26}$$

and estimate (25) is obtained by applying the Gronwall inequality to (26).

Let us show that the solution is unique. Let $z(\alpha)$ and $y(\alpha) \in Z$ be two solutions of (24). Since both of solutions lie in the ball $B_{r(\alpha)}$, $r(\alpha) = \|z_0\|\exp\Big(\frac{c_1T}{4\alpha}\Big)$, then according to Lemma 2 their difference $w(\alpha) = z(\alpha) - y(\alpha)$ satisfies the inequality

$$\|w(\alpha)\|(x) \le c_2(r(\alpha),\alpha,T)\int_0^x \|z - y\|(\xi)\,d\xi, \quad x \in [0, T/2],$$

which, for any $\alpha > 0$, has the unique solution $w(\alpha) = 0$, that is, $z(\alpha) = y(\alpha)$.

Let us prove the existence of the solution by the method of successive approximations. We have

$$z^{(n+1)}(\alpha) = z_0 + M(z^{(n)}(\alpha), \alpha), \quad n \geq 0, \quad z^{(0)} = z_0. \tag{27}$$

With inequality (22) and the principle of induction, it can be easily shown that for any $n \geq 0$:

$$\|z^{(n)}(\alpha)\|(x) \leq \|z_0\| \sum_{k=0}^{n} \frac{1}{k!}\Big(\frac{c_1 x}{2\alpha}\Big)^k \leq \|z_0\| \exp\Big(\frac{c_1 T}{4\alpha}\Big),$$

that is, all approximations lie in the ball $B_{r(\alpha)}$.

Consider the sequence $w^{(n)}(\alpha) = z^{(n+1)}(\alpha) - z^{(n)}(\alpha)$. We have

$$\begin{aligned}
\|w^{(0)}(\alpha)\| &= \|M(z_0, \alpha)\| \leq c_3\|z_0\|, \quad c_3 = \frac{c_1 T}{4\alpha}, \\
\|w^{(n)}(\alpha)\| &= \|M(z^{(n)}(\alpha), \alpha) - M(z^{(n-1)}(\alpha), \alpha)\| \\
&\leq c_2(r(\alpha), \alpha, T) \int_0^x \|w^{(n-1)}(\alpha)\|(\xi)\, d\xi, \quad n \geq 1.
\end{aligned}$$

Hence, for any $n \geq 0$

$$\|w^{(n)}(\alpha)\| \leq c_3\|z_0\| \frac{1}{n!}\left(\frac{c_2 T}{2}\right)^n.$$

It follows that the series $z_0 + \sum_{n=0}^{\infty} w^{(n)}(\alpha)$ is majorated by the convergent numerical series

$$\|z_0\| + c_3\|z_0\| \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{c_2 T}{2}\right)^n = \|z_0\|(1 + c_3 \mathrm{e}^{c_2 T/2}).$$

Hence, the sequence

$$z^{(n+1)}(\alpha) = z_0 + M(z^{(n)}(\alpha), \alpha) = z_0 + \sum_{m=0}^{n} w^{(m)}(\alpha)$$

converges in Z. Since all $z^{(n)}(\alpha) \in B_{r(\alpha)}$, we have

$$z(\alpha) = \lim_{n\to\infty} z^{(n)}(\alpha) \in B_{r(\alpha)}.$$

Going to the limit in (27), due to the continuity of the operator M, we find that z is the solution to equation (24).

The fact that the solution $z(\alpha)$ is continuously differentiable follows from the implicit function theorem [26]. Actually, according to Lemma 2, the operator $G : Z \times \bar{R}_+ \to Z$, $G(z, \alpha) = z - z_0 - M(z, \alpha)$, has the continuous derivatives $G_\alpha = -M_\alpha$ and $G_z = I - M_z$. In this case, for any $(z, \alpha) \in Z \times \bar{R}_+$, $G_z(z, \alpha) : Z \to Z$ has a bounded inverse operator (since $M_z(z, \alpha)$ is a linear continuous Volterra operator). Hence, according to the implicit function theorem, $z(\alpha) \in C^1(\bar{R}_+, Z)$. □

Consider the case, where $z_0 \in Z_0$. Then, according to Lemma 1, in Z there exists a unique solution $z(0)$ of the equation

$$z = z_0 + M(z, 0)$$

(the uniqueness of the solution follows from the uniqueness of the initial inverse problem [21, 22]). Hence, if $z_0 \in Z_0$, equation (24) is uniquely solvable in Z for any $\alpha \geq 0$. It is easy to see that in this case the solution $z(\alpha)$ will be continuously differentiable on the closed semi-axis $\bar{R}_+$. Actually, according to Theorem 1 it is sufficient to prove that $z(\alpha^2)$ is smooth in the neighborhood of the point $\alpha = 0$. This again follows from the implicit function theorem, since there exists a solution to the equation $\tilde{G}(z, 0) = 0$, $\tilde{G} \in C^1(Z \times R, Z)$, where $\tilde{G}(z, \alpha) = G(z, \alpha^2)$, and the operator $\tilde{G}_z(z(0), 0)$ has a bounded inverse. Let us formulate this result as a corollary of Theorem 1.

Corollary. *If $z_0 \in Z_0$, a solution to equation* (24) *exists and is unique in Z for all $\alpha \geq 0$, and belongs to the class in $C^1(\bar{R}_+, Z)$.*

Now, let us turn back to the initial problem. Thus, we know the function $\tilde{\phi} \in \tilde{\Phi}$ such that $\phi(0) = \tilde{\phi}(0)$, $\|\phi - \tilde{\phi}\| \leq \delta$, $\phi \in \Phi$. Hence, $z_0 = P\phi \in Z_0$ and $\tilde{z}_0 = P\tilde{\phi} \in \tilde{Z}_0$, $\|z_0 - \tilde{z}_0\| \leq \delta$.

Consider the equation

$$z = \tilde{z}_0 + M(z, \alpha).$$

According to Theorem 1, at $\alpha > 0$ it has a unique solution in Z. Let it be denoted by $\tilde{z}(\alpha)$. The solution to the equation

$$z = z_0 + M(z, \alpha), \quad \alpha \geq 0,$$

is denoted by $z(\alpha)$. We recall that $z(0)$ corresponds to the exact solution to the inverse problem. The function $\tilde{z}(\alpha)$ generates the operator $N : \tilde{Z}_0 \times \bar{R}_+ \to Z$, $N(\tilde{z}_0, \alpha) = \tilde{z}(\alpha)$. In the next theorem it is stated that this operator is regularizing for the equation $z = z_0 + M(z, 0)$.

Theorem 2. *Let $\delta \leq \delta_0$. Then there exists a function*

$$\alpha(\delta) \in C(0, \delta_0], \quad \alpha > 0, \quad \lim_{\delta \to 0} \alpha(\delta) = 0,$$

such that

$$\lim_{\delta \to 0} \|\tilde{z}(\alpha(\delta)) - z(0)\| = 0.$$

Proof. By virtue of the triangle inequality,

$$\|\tilde{z}(\alpha) - z(0)\| \leq \|\tilde{z}(\alpha) - z(\alpha)\| + \|z(\alpha) - z(0)\|.$$

Let $\alpha \le \alpha_0$, where the number α_0 will be specified later. According to the corollary of Theorem 1, the function $z(\alpha) \in C^1(\bar{R}_+, Z)$ and, hence, there exists a constant C_1 such that for all $\alpha \in [0, \alpha_0]$

$$\|z(\alpha) - z(0)\| \le C_1 \alpha.$$

Now, we estimate the difference $\tilde{z}(\alpha) - z(\alpha)$. We have

$$\|\tilde{z}(\alpha) - z(\alpha)\|(x) \le \|\tilde{z}_0 - z_0\| + \|M(\tilde{z}(\alpha), \alpha) - M(z(\alpha), \alpha)\|(x).$$

As in the proof of (23), it is easy to obtain the inequality

$$\|M(\tilde{z}(\alpha), \alpha) - M(z(\alpha), \alpha)\|(x) \le \left(\frac{c_1}{2\alpha} + \frac{2\|z(\alpha)\|}{\sqrt{\alpha}}\right) \int_0^x \|\tilde{z}(\alpha) - z(\alpha)\|(\xi)\, d\xi.$$

Note that the norms $\|z(\alpha)\|$ are uniformly bounded on $\bar{R}_+$ by some constant l depending on z_0. This follows from the continuity of the function $z(\alpha)$ on $\bar{R}_+$ and estimate (25). Thus,

$$\|\tilde{z}(\alpha) - z(\alpha)\|(x) \le \delta + \frac{C_2}{\alpha} \int_0^x \|\tilde{z}(\alpha) - z(\alpha)\|(\xi\,) d\xi, \quad x \in [0, T/2],$$

where $C_2 = (c_1 + 4l\sqrt{\alpha_0})/2$. Hence, by virtue of the Gronwall inequality, we obtain the estimate

$$\|\tilde{z}(\alpha) - z(\alpha)\|(x) \le \delta\, \mathrm{e}^{C_2 T/2\alpha}.$$

As a result, we have the inequality

$$\|\tilde{z}(\alpha) - z(0)\| \le C_1 \alpha + \delta\, \mathrm{e}^{C_2 T/(2\alpha)}.$$

Now, it is sufficient to take

$$\alpha(\delta) = \frac{C_1 + 4l\sqrt{\alpha_0}}{1 + \ln(\delta_0/\delta)} \frac{T}{2}.$$

Here

$$\alpha_0 = \left(Tl + \sqrt{T^2 l^2 + C_1 T/2}\right)^2 = \alpha(\delta_0).$$

Then

$$\|\tilde{z}(\alpha(\delta)) - z(0)\| \le C_1 \alpha(\delta) + \sqrt{e\delta_0\delta} \to 0, \quad \delta \to 0. \qquad \square$$

It follows from the proof of the theorem that the operator N will be a uniformly regularizing operator on any subset Z_{0l} of the set Z_0 of the form $Z_{0l} = \{z_0 \in Z_0 : \|z_0\| < l\}$.

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