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Spherical mean value theorem for a poroelastic static system

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Abstract. Mean value relations for a vector of displacement of an elastic porous body and a pore pressure for a poroelastic static system, when mass forces and energy dissipation are absent, are obtained.

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1. Introduction

It is well-known, what an important role in mathematical physics is played by classical mean value theorems for harmonic analysis [1]. Mean value relations are most useful, also, in computational mathematics as they give an effective method of constructing difference schemes. In Monte Carlo methods, mean value theorems play a special role as they are basic for constructing algorithms of a random walk on spheres [2–4]. In [4, 5], mean value relations for a system of Lame equations and thermoelasticity are obtained.

Simulation of two phase flows in heterogenous porous media is widely used in oil production. For example, the simulation of a reservoir is intended reconstructing a geological history of a sedimentary basin and, in particular, dislocation of a hydrocarbon component on a geological time scale. The simulation of a reservoir deals with understanding and prediction of fluid flows occurring in the processes of oil production. On the other hand, the simulation of two phase flows in porous media plays an important role for the prognosis of earthquakes preparation as this is an energy intensive process. Opening cracks in the zones with increased values of the shearing and the tensile stresses is the most universal mechanism of development of changes in a porous medium. Such zones are formed in the vicinity of the sources of future earthquakes if a distribution of forces in space is non uniform. Many seismologists consider that the initial stage of opening cracks and a subsequent state of the medium, when destruction processes develop, is associated with the dilatancy of the medium described in [6–8].

Dilatancy is a nonlinear loosening of a medium due to the formation of cracks caused by a shear. This takes place, when tangential stresses exceed a certain threshold. A dilatancy area is considered to incorporate a set of elastic porous medium points, for which at a given stress field $\{\sigma_{ij}\}$ the following condition is fulfilled:

$$D_\tau \equiv \tau - \alpha(P + \rho g z) - Y \geq 0, \tag{1}$$

where ρ is the density of rocks, g is the acceleration of gravity, z is the depth of a point, P is the hydrodynamic pressure, α is the internal friction coefficient, Y is the cohesion of rocks, τ is the intensity of tangential stresses:

$$\tau = \frac{\sqrt{3}}{2}\Big[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{11} - \sigma_{33})^2 + 6(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2)\Big]^{1/2}.$$

Condition (1) coincides with the Schleicher–Nadai criterion of material destruction under the action of shearing loads. It satisfactorily describes the beginning of the rock destruction process. It can also be used at the "pre-destruction" stage (when loading constitutes up to 60–90 % of the critical value) for the qualitative description of the shape of areas with intensification of crack opening.

In the given paper, mean value relations for equations of a poroelastic static system are obtained using the proposed method [4, 5]. Namely, the mean value relations for a vector of displacement of an elastic porous body and a pore pressure are obtained. The knowledge of these values is sufficient, on the one hand, for evaluation of reservoirs in oil production and, on the other hand, for definition of a dilatancy area in problems of the prognosis of earthquakes.

2. Statement of the problem

Let us assume that the bounded domain $\widetilde{\Omega} \subset R^3$ is filled with a homogeneous isotropic elastic porous medium. The elastic porous static state of the medium $\widetilde{\Omega}$ in the absence of mass sources and dissipation of energy is described by the system of the differential equations [9, 10]:

$$\frac{\rho_{0,s}}{\rho_0}\frac{\partial P}{\partial x_i} + \sum_{k=1}^{3}\frac{\partial \bar{h}_{ik}}{\partial x_k} = 0, \qquad \frac{\rho_{0,l}}{\rho_0}\frac{\partial P}{\partial x_i} = 0, \quad i = 1, 2, 3. \tag{2}$$

Here $\bar{h}_{ik}$ is a stress tensor, P is the pore pressure, $\rho_0 = \rho_{0,l} + \rho_{0,s}$, $\rho_{0,l}$ and $\rho_{0,s}$ are partial densities of fluid and an elastic porous body, respectively. The total stress tensor of the elastic porous body looks like

$$\sigma_{ik} = -\bar{h}_{ik} - P\delta_{ik}, \tag{3}$$

$$\bar{h}_{ik} = -\mu\bigg(\frac{\partial U_i}{\partial x_k} + \frac{\partial U_k}{\partial x_i}\bigg) - \bigg(\lambda - \frac{\rho_{0,s}}{\rho_0}K\bigg)\delta_{ik}\,\mathrm{div}\,\boldsymbol{U} + \frac{\rho_{0,l}}{\rho_0}K\delta_{ik}\,\mathrm{div}\,\boldsymbol{V}, \tag{4}$$

$$P=\left(K-\left(\rho_0^2\hat{\alpha}+\frac{K}{\rho_0}\right)\rho_{0,s}\right)\operatorname{div}\boldsymbol{U}-\left(\rho_0^2\hat{\alpha}+\frac{K}{\rho_0}\right)\rho_{0,l}\operatorname{div}\boldsymbol{V},\qquad(5)$$

where $\boldsymbol{U}=(U_1,U_2,U_3)$ is the vector of displacement of the elastic porous body, $\boldsymbol{V}=(V_1,V_2,V_3)$ is the vector of displacement of fluid, $K=\lambda+\frac{2}{3}\mu$, δ_{ik} is the Kronecker delta, and $\hat{\alpha}$, λ, μ are the constants from the equation of state [9].

3. Differential equations of the vector of displacement of an elastic porous body and pore pressure

Using the pore pressure definition (5), we exclude from (4) the vector of displacement of fluid. The obtained expression is substituted into the first equation of system (2), and the operator div will act on both sides of the second equation of system (2). As a result, we will obtain a system of second order differential equations of the vector of displacement of the elastic porous body $\boldsymbol{U}$ and the pore pressure P:

$$\Delta\boldsymbol{U}+\tilde{\alpha}\nabla\operatorname{div}\boldsymbol{U}+\tilde{\beta}\nabla P=0,\qquad \Delta P=0.\qquad(6)$$

Here,

$$\tilde{\alpha}=1+\frac{\lambda}{\mu}-\frac{K^2}{(\rho_0^3\hat{\alpha}+K)\mu},\qquad \tilde{\beta}=\frac{K\rho_{0,l}-\rho_0^3\hat{\alpha}\rho_{0,s}}{(\rho_0^3\hat{\alpha}+K)\mu\rho_0}.$$

Now, following [4, 5], introduce $N(\boldsymbol{u})$ and $N^{(1)}(\boldsymbol{u})$, which are operators of averaging the vector function $\boldsymbol{u}=(u_1,u_2,\dots,u_n)^T$ on the surface of the sphere $S(\boldsymbol{x},r)$ on the uniform measure $d\Omega$ and $d\eta^{(1)}=\{[a\delta_{ij}+bs_is_j]d\Omega\}_{i,j=1}^n$, respectively, i.e.,

$$N(\boldsymbol{u})=\frac{1}{\omega_n r^{n-1}}\int_{S(\boldsymbol{x},r)}\boldsymbol{u}(\boldsymbol{x}+r\boldsymbol{y})\,d\Omega(\boldsymbol{y}),$$
$$N^{(1)}(\boldsymbol{u})=\frac{1}{\omega_n r^{n-1}}\int_{S(\boldsymbol{x},r)}\boldsymbol{u}(\boldsymbol{x}+r\boldsymbol{y})\,d\eta^{(1)}(\boldsymbol{y}),$$

where ω_n is the area of the unit sphere, s_i are the direction cosines, a and b are known constants.

For a harmonic function $P(\boldsymbol{x})$, $\boldsymbol{x}\in\widetilde{\Omega}$, we have

$$P(\boldsymbol{x})=N(P)(\boldsymbol{x})=\frac{3}{4\pi r^3}N^{(W)}P(\boldsymbol{x}),\qquad(7)$$

where $N^{(W)}P(\boldsymbol{x})$ is a volume integral of P on $W=\{|\boldsymbol{x}-\boldsymbol{y}|<r\}$. For the harmonic function $\dfrac{\partial P}{\partial x_k}$, using (7) we obtain

$$\frac{\partial P(\boldsymbol{x})}{\partial x_k} = \frac{3}{4\pi r^3}\int_W \frac{\partial P}{\partial x_k}\,dW = \frac{3}{4\pi r^3}\int_{S(\boldsymbol{x},1)} P\frac{x_k}{r}\,d\Omega, \quad k=1,2,3,$$

or

$$\nabla P(\boldsymbol{x}) = \frac{3}{4\pi r^3}\int_{S(\boldsymbol{x},1)} P\nabla s\,d\Omega(s). \tag{8}$$

Now, using the formula from [5] and taking into account a biharmonic vector of displacement of the elastic porous body $\boldsymbol{U}$ and then relation (8), we obtain

$$\begin{aligned} &N^{(1)}(\boldsymbol{U})(\boldsymbol{x}) - \boldsymbol{U}(\boldsymbol{x}) \\ &\quad= \frac{r^2(\Delta + \tilde\alpha\nabla\,\mathrm{div})\boldsymbol{U}(\boldsymbol{x})}{\tilde\alpha+3} + \\ &\quad\quad \sum_{n=2}^{\infty}\frac{2(n+1)r^{2n}}{(2n+3)!}\left[\frac{15n\Delta^{n-1}(-\tilde\beta)\nabla P(\boldsymbol{x})}{\tilde\alpha+3} - 3(n-1)\Delta\boldsymbol{U}(\boldsymbol{x})\right] \\ &\quad= -\frac{\tilde\beta r^2}{\tilde\alpha+3}\nabla P(\boldsymbol{x}) = -\frac{\tilde\beta r^2}{\tilde\alpha+3}\frac{3}{4\pi r^3}\int_{S(\boldsymbol{x},1)} P\nabla s\,d\Omega(s). \end{aligned}$$

Consequently, taking into account the harmonicity of P and the biharmonicity of $\boldsymbol{U}$, we have proved the following theorem:

Theorem. *The general solution to the system of equations* (6) *of the class* $C^\infty(\widetilde{\Omega})$ *satisfies the mean value relation*

$$P(\boldsymbol{x}) = N(P)(\boldsymbol{x}), \quad \boldsymbol{U}(\boldsymbol{x}) = N^{(1)}(\boldsymbol{U})(\boldsymbol{x}) + \frac{\tilde\beta r^2}{\tilde\alpha+3}\frac{3}{4\pi r^3}\int_{S(\boldsymbol{x},1)} P\nabla s\,d\Omega(s).$$

Components of the stress tensor are calculated by the formula from [11],

$$\sigma_{ik} = \mu\left(\frac{\partial U_i}{\partial x_k} + \frac{\partial U_k}{\partial x_i}\right) + \left(\lambda - \frac{K^2}{K+\rho_0^3\hat\alpha}\right)\delta_{ik}\,\mathrm{div}\,\boldsymbol{U} - \frac{\rho_0^3\hat\alpha}{K+\rho_0^3\hat\alpha}\delta_{ik}P,$$

if the vector of displacement of the elastic porous body $\boldsymbol{U}$ *and the pore pressure* P *are known.*

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