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The timed barbed bisimulation is decidable for timed transition systems with invariants*

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Abstract. Timed transition systems are a widely studied model for real-time systems. In this work we deal with an extension of this model, timed transition systems with invariants. The intention of the paper is to show applicability of the general categorical framework of open maps in order to treat the decidability question of the timed barbed bisimulation in the setting of the model being studied.

In particular, we define a category of timed transition systems with invariants, whose morphisms are to be considered as simulations of the behavior of one system by the other with an accuracy of τ-actions, and an accompanying (sub)category of path objects, for which the corresponding notions of open maps are developed. We then use the open maps framework to obtain the abstract bisimilarity and the path bisimilarity which are established to coincide with the timed barbed bisimulation. Finally, we consider the decidability question of the bisimulation studied in the setting of finite timed transition systems with invariants.

1. Introduction

Timed automata are a useful tool for modeling real-time systems. A theory of timed automata and timed languages has emerged in the past 15 years, leading to generalizations of the classical results for non-timed automata on regular expressions, algebraic characterizations and logics. The paper [3] accounts for an interesting discussion on challenges that remain in order to provide good generalizations of some important results from the classical automata theory. The central component of the theory of timed automata is timed versions of equivalences on concurrent real-time processes. Loosely speaking, for two processes to be timed equivalent they should agree not only on what actions they can perform, but on when these actions are performable.

In an attempt to explain and unify apparent differences within the extensive amount of research in the field of untimed behavioural equivalences, several category-theoretic approaches to the matter appeared. One of them was initiated by Joyal, Nielsen, and Winskel in [15] where they have proposed abstract ways of capturing the notion of bisimulation through open maps based bisimilarity and its logical counterpart — path bisimilarity. As shown in [15, 19, 20], bisimilarity induced by open maps makes possible a

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uniform definition of the numerous suggested behavioural equivalences (e.g., trace and testing equivalences, bisimulation, barbed and weak bisimulation, etc.) across a wide range of models of concurrency (e.g., transition systems, event structures, Petri nets, higher dimensional automata, etc.). This approach was successfully generalized to timed automata (see [13], [24]), represented by the most popular and well-studied model — the timed transition systems. In the paper, we deal with an extension of this model, timed transition systems with invariants on the states, and try to extended the existing category-theoretic results to a timed variant of Milner and Sangiorgi's barbed bisimulation. The goal of the paper is to provide open maps and logical characterizations of bisimulation studied and to use them in order to treat the decidability question of this bisimilarity in the setting of finite timed transition systems with invariants.

Milner and Sangiorgi's barbed bisimilarity [18] is a widely used concurrency semantics for process algebras that include the silent step τ. It differs from Milner's (strong) bisimulation in the following: 'visible' and 'invisible' actions are distinguished; transitions labelled by 'invisible' actions are required to be bisimulated; only the existence of a transition labelled by a 'visible' action has to be matched. An important feature of the barbed bisimulation is that it can be successfully employed when the operational semantics of a process algebra is defined by a reduction relation (i.e., no labels over transitions). It allows one to recover from such a formulation the well-known bisimulation-based equivalences which are defined on the labelled transition system. Another advantage of the barbed bisimulation semantics is that it can be defined uniformally in different processes calculi (e.g., CCS, π-calculus, higher order π-calculus). See [18] for a clear account on all strong points of the barbed bisimilarity. In [6] several timed variants of the barbed bisimulation have been introduced and their distinguishing power over processes have been studied in the context of π-calculus with locations, types and timers.

The rest of the paper is organized as follows. The basic notions and notations related to timed transition systems with invariants and their behaviour are introduced in Section 2. A definition of the timed version of Milner and Sangiorgi's barbed bisimulation is given in Section 3. In Section 4, we give the basic elements of the category theory and provide the category-theoretic characterization for the studied bisimulation. In the next section, we show how the bisimulation under consideration can be captured by another category-theoretic bisimilarity — path-bisimilarity. The decidability question of the timed barbed bisimulation in the setting of finite transition systems with invariants is treated in Section 6. Section 7 contains conclusion and some remarks on future work.

2. Timed transition systems with invariants

In this section, we define some basic notions concerning the structure and behaviour of timed transition systems with invariants [13].

Before doing so, it will be convenient to introduce some auxiliary notions and notations. Let $\mathbf{R}$ be the set of non-negative reals. Also, let Σ be a finite alphabet of actions without the *silent* action τ, and $\Sigma_\tau = \Sigma \cup \{\tau\}$. A *timed word over* Σ_τ is a finite sequence of pairs $\alpha = (\sigma_1, d_1) \dots (\sigma_n, d_n)$ such that $\sigma_i \in \Sigma_\tau$, $d_i \in \mathbf{R}$, for all $1 \le i \le n$, and $d_i < d_{i+1}$ for all $1 \le i < n$. A pair (σ_i, d_i) represents an occurrence of an action σ_i at time d_i relative to the starting time (0) of the execution. Let ε denote the empty timed word. We consider a finite set V of clock variables. A *clock valuation over* V is a mapping $\nu : V \to \mathbf{R}$ which assigns time values to the clock variables of a system. Define $(\nu + c)(x) := \nu(x) + c$ for all clock variables $x \in V$ and constants $c \in \mathbf{R}$. For a subset λ of clock variables, we shall write $\nu[\lambda \to 0](x) := 0$, if $x \in \lambda$, and $\nu[\lambda \to 0](x) := \nu(x)$, otherwise. Given a set V, we define the set $\Delta(V)$ of *clock constraints* by the following grammar: $\delta ::= c \# x \mid x + c \# y \mid \delta \wedge \delta$, where $\# \in \{\le, <, \ge, >, =\}$, c is a real valued constant and x, y are clock variables from V. We shall say that a clock constraint δ is *satisfied by a clock valuation* ν if the expression $\delta[\nu(x)/x]$[1] evaluates to true. A clock constraint δ defines a subset of $\mathbf{R}^m$ (m is the number of clock variables in V), which is called the meaning of δ and denoted by $\|\delta\|_V$. A clock valuation ν defines a point in $\mathbf{R}^m$ (denoted by $\|\nu\|_V$). So, the clock constraint δ is satisfied by the clock valuation ν iff $\|\nu\|_V \in \|\delta\|_V$.

We are now prepared to consider the definition of timed transition systems.

Definition 1. A *timed transition system with invariants* $\mathcal{T}$ over an alphabet Σ_τ is a sextuple $(S, s_0, \Sigma_\tau, V, T, I)$, where

- S is a set of states with the initial state s_0,
- V is a set of clock variables,
- $T \subseteq S \times \Sigma_\tau \times \Delta(V) \times 2^V \times S$ is a set of transitions,
- $I \in \Delta(V)^S$ assigns to each state an invariant given by the same syntax as that of a clock constraint.

We shall write $s \underset{\delta,\ \lambda}{\overset{\sigma}{\to}} s'$ to denote a transition $(s, \sigma, \delta, \lambda, s')$.

Define the behaviour of timed transition systems with invariants.

[1] $\delta[y/x]$ is the substitution of y for x in δ.

Definition 2. Let $\mathcal{T}$ be a timed transition system with invariants over an alphabet Σ_τ.

A *configuration* of $\mathcal{T}$ is a pair $\langle s, \nu \rangle$, where s is a state and ν is a clock valuation. A configuration $\langle s, \nu \rangle$ of $\mathcal{T}$ is called *initial* iff s is the initial state and ν is the constant 0 function.

A *run* of $\mathcal{T}$ is a sequence $\gamma = \langle s_0, \nu_0 \rangle \xrightarrow[d_1]{\sigma_1} \langle s_1, \nu_1 \rangle \ldots \langle s_{n-1}, \nu_{n-1} \rangle \xrightarrow[d_n]{\sigma_n} \langle s_n, \nu_n \rangle$ such that $\langle s_0, \nu_0 \rangle$ is the initial configuration, $\|\nu_n\|_V \in \|I(s_n)\|_V$ and for all $0 < i \leq n$ and $0 \leq d \leq (d_i - d_{i-1})$ it holds that $\|\nu_{i-1} + d\|_V \in \|I(s_{i-1})\|_V$ and there is a transition $s_{i-1} \xrightarrow[\delta_i,\ \lambda_i]{\sigma_i} s_i$ such that $\|\nu_{i-1} + (d_i - d_{i-1})\|_V \in \|\delta_i\|_V$ and $\nu_i = (\nu_{i-1} + (d_i - d_{i-1}))[\lambda_i \to 0]$. Here, d_0 is defined to be 0. We will use $Runs(\mathcal{T})$ to denote the set of runs of $\mathcal{T}$. A run γ as above is said to *generate the timed word* $\alpha = (\sigma_1, d_1) \ldots (\sigma_n, d_n)$.

A configuration $\langle s, \nu \rangle$ of $\mathcal{T}$ is called *reachable* iff $\mathcal{T}$ has a run with an occurrence of $\langle s, \nu \rangle$. The set of reachable configurations of $\mathcal{T}$ is denoted as $Conf(\mathcal{T})$.

A state $s \in S$ is called τ-*accessible* iff $s_0 \xrightarrow[\delta_1,\lambda_1]{\tau} s_1 \ldots s_{n-1} \xrightarrow[\delta_n,\lambda_n]{\tau} s_n = s$ $(n \geq 0)$. Define the set $S_\tau(\mathcal{T}) = \{s \in S \mid s \text{ is a } \tau\text{-accessible state}\}$.

A configuration $\langle s, \nu \rangle$ of $\mathcal{T}$ is called τ-*reachable* if there is a run $\langle s_0, \nu_0 \rangle \xrightarrow[d_1]{\tau} \langle s_1, \nu_1 \rangle \ldots \langle s_{n-1}, \nu_{n-1} \rangle \xrightarrow[d_n]{\tau} \langle s_n, \nu_n \rangle = \langle s, \nu \rangle$ generating the timed word $(\tau, d_1) \ldots (\tau, d_n)$. We shall use $Conf_\tau(\mathcal{T})$ to denote the set of τ-reachable configurations.

A τ-reachable configuration $\langle s, \nu \rangle$ of $\mathcal{T}$ is called *final* iff there is no run ends in $\langle s, \nu \rangle$ that can be extended by some τ-timed transition.

From now on, for a configuration $\langle s, \nu \rangle \in Conf_\tau(\mathcal{T})$, we shall write $\langle s, \nu \rangle \xrightarrow[d]{\sigma}$ iff there is a configuration $\langle s', \nu' \rangle \in Conf(\mathcal{T})$ such that

$$\langle s, \nu \rangle \xrightarrow[d]{\sigma} \langle s', \nu' \rangle,$$

for some $\sigma \in \Sigma$ and $d \in \mathbf{R}$.

Example 1. To illustrate the concepts, consider the timed transition system $\mathcal{T}$ over Σ_τ (see Fig. 1) which has three states s_0 (the initial state), s_1 and s_2, with invariants $x \leq 6 \wedge y \leq 4$, $3 \leq x \leq 8$ and $x \leq 6$, respectively, two actions a and τ, and two clock variables x and y. Three transitions depicted between the states are labeled by actions, clock constraints and subsets of clocks. For instance, one of the transitions between s_0 and s_1 is labelled by an action τ, a clock constraint $x = 3$ and a subset $\{y\}$ of clock variables. The timed transition system $\mathcal{T}$ has the only τ-accessible state s_1. Consider the run $\langle s_0, \nu_0 \rangle \xrightarrow[3]{\tau} \langle s_1, \nu_1 \rangle \xrightarrow[6]{a} \langle s_0, \nu_2 \rangle$ with $\nu_1(x) = 3$, $\nu_1(y) = 0$, $\nu_2(x) = 0$

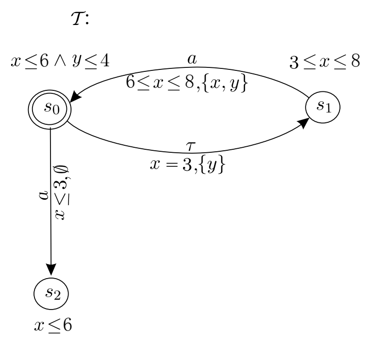

Figure 1

and $\nu_2(y) = 0$. The run generates the timed word $(\tau, 3)\,(a, 6)$. Moreover, $\langle s_1, \nu_1 \rangle$ is a τ-reachable configuration, but $\langle s_0, \nu_2 \rangle$ is not.

3. Timed barbed bisimulation

In this section, we define a timed extension of Milner and Sangiorgi's barbed bisimulation [18].

Definition 3. Timed transition systems $\mathcal{T}$ and $\mathcal{T}'$ over an alphabet Σ_τ are *timed barbed bisimilar* iff there is a relation $\mathcal{B} \subseteq Conf_\tau(\mathcal{T}) \times Conf_\tau(\mathcal{T}')$ such that $(\langle s_0, \nu_0 \rangle, \langle s'_0, \nu'_0 \rangle) \in \mathcal{B}$ and for all $(\langle s, \nu \rangle, \langle s', \nu' \rangle) \in \mathcal{B}$ it holds:

- if $\langle s, \nu \rangle \xrightarrow[d]{\tau} \langle s_1, \nu_1 \rangle$ in $\mathcal{T}$, then $\langle s', \nu' \rangle \xrightarrow[d]{\tau} \langle s'_1, \nu'_1 \rangle$ in $\mathcal{T}'$ and $(\langle s_1, \nu_1 \rangle, \langle s'_1, \nu'_1 \rangle) \in \mathcal{B}$, for some $\langle s'_1, \nu'_1 \rangle$,
- if $\langle s', \nu' \rangle \xrightarrow[d]{\tau} \langle s'_1, \nu'_1 \rangle$ in $\mathcal{T}'$, then $\langle s, \nu \rangle \xrightarrow[d]{\tau} \langle s_1, \nu_1 \rangle$ in $\mathcal{T}$ and $(\langle s_1, \nu_1 \rangle, \langle s'_1, \nu'_1 \rangle) \in \mathcal{B}$, for some $\langle s_1, \nu_1 \rangle$,
- if $\langle s, \nu \rangle \xrightarrow[d]{\sigma}$ $(\sigma \in \Sigma)$ in $\mathcal{T}$, then $\langle s', \nu' \rangle \xrightarrow[d]{\sigma'}$ $(\sigma' \in \Sigma)$ in $\mathcal{T}'$,
- if $\langle s', \nu' \rangle \xrightarrow[d]{\sigma'}$ $(\sigma' \in \Sigma)$ in $\mathcal{T}'$, then $\langle s, \nu \rangle \xrightarrow[d]{\sigma}$ $(\sigma \in \Sigma)$ in $\mathcal{T}$.

Example 2. We illustrate the notions and notations related to the definition of the timed barbed bisimulation. Consider the timed transition systems in Fig. 2. It is easy to see that the timed transition systems $\overline{\mathcal{T}}$ and $\widetilde{\mathcal{T}}$ are timed barbed bisimilar, while the timed transition systems $\widetilde{\mathcal{T}}$ and $\check{\mathcal{T}}$ are not, because, for example, after the occurrence of an action τ at time 1, the occurrence of an action τ at time 5 is possible in $\check{\mathcal{T}}$ but it is not the case in $\widetilde{\mathcal{T}}$. $\diamondsuit$

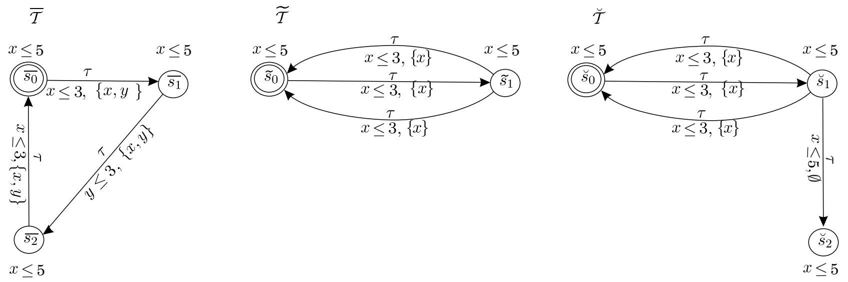

Figure 2

4. Open maps bisimulation

4.1. A basic elements of category theory

In this section, we briefly recall the basic definitions from the category theory.

The category theory is a generalized mathematical theory of structures. One of its goals is to reveal the universal properties of structures of a given kind (objects of a category) via their mutual relationships (morphisms of a category). From the 1980s to this day, the category-theoretic methods found many applications in theoretical computer science.

A *category* $\mathbb{M}$ consists of the following:

- a set $\mid \mathbb{M} \mid$ whose elements are called the *objects*,
- for every pair X and Y of objects, a set $\mathbb{M}(X, Y)$ whose elements are called the *morphisms* from X to Y,
- for every triple X, Y and Z of objects, a *composition* law: $\circ : \mathbb{M}(X, Y) \times \mathbb{M}(Y, Z) \longrightarrow \mathbb{M}(X, Z)$; the composition of $f \in \mathbb{M}(X, Y)$ and $g \in \mathbb{M}(Y, Z)$ is written as $g \circ f$,
- for every object X, a morphism $1_X \in \mathbb{M}(X, X)$ is called the *identity* on X,

Furthermore, morphisms have to satisfy two axioms:

1. *Associativity:* given the morphisms $f \in \mathbb{M}(X, Y)$, $g \in \mathbb{M}(Y, Z)$ and $h \in \mathbb{M}(Z, V)$, the following equality holds: $h \circ (g \circ f) = (h \circ g) \circ f$,
2. *Identity:* given the morphisms $f \in \mathbb{M}(X, Y)$ and $g \in \mathbb{M}(Y, Z)$, the following equalities hold: $1_Y \circ f = f$ and $g \circ 1_Y = g$.

It is sometimes useful if $\mathbb{M}$ has the following property. Consider two morphisms $X_1 \stackrel{m_1}{\to} X$ and $X_2 \stackrel{m_2}{\to} X$ in a category $\mathbb{M}$. A *pullback* of (m_1, m_2)

is a triple (X', m_1', m_2'), where X' is an object in $\mathbb{M}$ and $X' \stackrel{m_1'}{\to} X_1$ and $X' \stackrel{m_2'}{\to} X_2$ are morphisms in $\mathbb{M}$ such that:

- $m_1 \circ m_1' = m_2 \circ m_2'$,
- for any other triple (X'', m_1'', m_2''), where X'' is an object in $\mathbb{M}$ and $X'' \stackrel{m_1''}{\to} X_1$ and $X'' \stackrel{m_2''}{\to} X_2$ are morphisms in $\mathbb{M}$ such that $m_1 \circ m_1'' = m_2 \circ m_2''$, there exists a unique morphism $\widetilde{m} : X'' \to X'$ such that $m_1'' = m_1' \circ \widetilde{m}$ and $m_2'' = m_2' \circ \widetilde{m}$.

The concept of an open map (open morphism) appears in the paper by Joyal and Moerdijk [14], where the concept of a subcategory of open maps of a (pre)topos is defined. As reported in [15], the open map approach provides general concepts of bisimilarity for any categorical model of computation.

First, a category $\mathbb{M}$ whose objects represent models of computations has to be identified. A morphism $m : X \longrightarrow Y$ in $\mathbb{M}$ should intuitively be thought of as a simulation of the object X in the object Y. Then, inside the category $\mathbb{M}$, we choose a subcategory of 'path objects' and 'path extension' morphisms between these objects. The *subcategory of path objects* is denoted by $\mathbb{P}$. Given a path object P in $\mathbb{P}$ and a model object X in $\mathbb{M}$, a *path* is a morphism $p : P \longrightarrow X$ in $\mathbb{M}$. We think of p as representing a particular way of realizing P in X.

Second, we identify morphisms $m : X \longrightarrow Y$ which have the property that, whenever a computation of X can be extended via m in Y, that extension can be matched by an extension of the computation in X. A morphism $m : X \to Y$ in $\mathbb{M}$ is called $\mathbb{P}$-*open* if, whenever $f : P_1 \to P_2$ in $\mathbb{P}$, $p : P_1 \to X$ and $q : P_2 \to Y$ in $\mathbb{M}$, and the diagram

$$\begin{tikzcd} P_1 \arrow[r, "p"] \arrow[d, "f"'] & X \arrow[d, "m"] \\ P_2 \arrow[r, "q"'] & Y \end{tikzcd}$$

commutes, i.e. $m \circ p = q \circ f$, there exists a morphism $h : P_2 \to X$ in $\mathbb{M}$ such that the two triangles in the diagram

$$\begin{tikzcd} P_1 \arrow[r, "p"] \arrow[d, "f"'] & X \arrow[d, "m"] \\ P_2 \arrow[r, "q"'] \arrow[ur, "h"] & Y \end{tikzcd}$$

commute, i.e. $p = h \circ f$ and $q = m \circ h$.

Third, an abstract notion of bisimilarity is introduced. The definition is given in terms of spans of open maps. Two objects X and Y in $\mathbb{M}$ are said to be $\mathbb{P}$-*bisimilar*, if there exists a span $X \xleftarrow{m} Z \xrightarrow{m'} Y$ with a common object Z of $\mathbb{P}$-open morphisms.

Notice that if $\mathbb{M}$ has pullbacks, it can be shown that $\mathbb{P}$-bisimilarity is always an equivalence relation. The important observation is that pullbacks of open maps are open maps themselves [15].

4.2. Open maps characterization

In this section, a category of timed transition systems with invariants is introduced and an open maps based characterization of the timed barbed bisimulation is given.

The morphisms of our category will represent some notions of simulation of the behavior of one system by the other with an accuracy of τ-actions and with account of only the existence of visible actions. This leads to the following definition of a morphism consisting of two functions, one mapping τ-accessible states of the simulated system to simulating τ-accessible states of the other, and one mapping clocks of the simulating system to simulated clocks of the other.

Definition 4. Let $\mathcal{T} = (S, s_0, \Sigma_\tau, V, T, I)$ and $\mathcal{T}' = (S', \Sigma_\tau, s_0', V', T', I')$ be timed transition systems with invariants. A pair (μ, η) is called a *morphism* between $\mathcal{T}$ and $\mathcal{T}'$ iff $\mu : S_\tau(\mathcal{T}) \to S_\tau(\mathcal{T}')$ is a mapping between τ-accessible states and $\eta : V' \to V$ is a mapping between clock variables, which should satisfy the following:

- $\mu(s_0) = s_0'$,
- $\|I(s)\|_V \subseteq \|I'(\mu(s))[\eta(x)/x]\|_V$,
- if there is a transition $s \xrightarrow[\delta,\ \lambda]{\tau} s_1$ in $\mathcal{T}$, then there exists a transition $\mu(s) \xrightarrow[\delta',\lambda']{\tau} \mu(s_1)$ in $\mathcal{T}'$, such that $\|\delta\|_V \subseteq \|\delta'[\eta(x)/x]\|_V$ and $\lambda' = \eta^{-1}(\lambda)$,
- if there is a transition $s \xrightarrow[\delta,\ \lambda]{\sigma} s_1$ $(\sigma \in \Sigma)$ in $\mathcal{T}$, then there exists a transition $\mu(s) \xrightarrow[\delta',\lambda']{\sigma'} s_1'$ in $\mathcal{T}'$ $(\sigma' \in \Sigma)$ such that $\|I(s_1)\|_V \subseteq \|I'(s_1')[\eta(x)/x]\|_V$, $\|\delta\|_V \subseteq \|\delta'[\eta(x)/x]\|_V$ and $\lambda' = \eta^{-1}(\lambda)$.

Consider a useful property of a morphism. First, introduce an auxiliary notation. For a function $\eta : V' \to V$ and a clock valuation $\nu : V \to \mathbf{R}$, we define $\eta^{-1}(\nu) : V' \to \mathbf{R}$ as follows: $\eta^{-1}(\nu)(x') := \nu(\eta(x'))$.

Lemma 1. *Let (μ, η) be a morphism between timed transition systems with invariants $\mathcal{T}$ and $\mathcal{T}'$ over Σ_τ. If $\langle s_0, \nu_0\rangle \xrightarrow[d_1]{\tau} \langle s_1, \nu_1\rangle \ \ldots \ \langle s_{n-1}, \nu_{n-1}\rangle$*

$\xrightarrow[d_n]{\tau} \langle s_n, \nu_n\rangle$ *is a run in* $\mathcal{T}$ *generating the timed word* $(\tau, d_1)\ldots(\tau, d_n)$, *then* $\langle\mu(s_0), \eta^{-1}(\nu_0)\rangle \xrightarrow[d_1]{\tau} \langle\mu(s_1), \eta^{-1}(\nu_1)\rangle \ldots \langle\mu(s_{n-1}), \eta^{-1}(\nu_{n-1})\rangle \xrightarrow[d_n]{\tau} \langle\mu(s_n), \eta^{-1}(\nu_n)\rangle$ *is a run in* $\mathcal{T}'$ *generating the same timed word. Moreover, if* $\langle s_n, \nu_n\rangle \xrightarrow[d]{\sigma}$ $(\sigma \in \Sigma)$ *in* $\mathcal{T}$, *then* $\langle\mu(s_n), \eta^{-1}(\nu_n)\rangle \xrightarrow[d]{\sigma'}$ $(\sigma' \in \Sigma)$ *in* $\mathcal{T}'$.

Timed transition systems with the alphabet Σ_τ and morphisms between them form a category of timed transition systems, $\mathcal{TTSI}_{barbed}$, in which the composition of two morphisms $(\mu, \eta) : \mathcal{T} \longrightarrow \mathcal{T}'$ and $(\mu', \eta') : \mathcal{T}' \longrightarrow \mathcal{T}''$ is defined as $(\mu', \eta') \circ (\mu, \eta) := (\mu' \circ \mu, \eta \circ \eta')$, and the identity morphism is the morphism where both μ and η are the identity functions. The next theorem establishes an important property of the category $\mathcal{TTSI}_{barbed}$.

Theorem 1. $\mathcal{TTSI}_{barbed}$ *has pullbacks.*

Proof. Suppose that
$\mathcal{T}_0 = (S^0,\ \Sigma_\tau,\ s_0^0,\ V^0,\ T^0,\ I^0)$,
$\mathcal{T}_1 = (S^1,\ \Sigma_\tau,\ s_0^1,\ V^1,\ T^1,\ I^1)$,
$\mathcal{T}_2 = (S^2,\ \Sigma_\tau,\ s_0^2,\ V^2,\ T^2,\ I^2)$
are timed transition systems with invariants. Also, assume $\mathcal{T}_1 \overset{(\mu_1,\eta_1)}{\longrightarrow} \mathcal{T}_0 \overset{(\mu_2,\eta_2)}{\longleftarrow} \mathcal{T}_2$ to be a construction of morphisms in the category $\mathcal{TTSI}_{barbed}$. Construct a timed transition system $\mathcal{T} = (S, s_0, \Sigma_\tau, V, T, I)$ with two morphisms $(\mu_i', \eta_i') : \mathcal{T} \to \mathcal{T}_i$ $(i = 1, 2)$ as follows:

- $S = S' \cup S''$, where $S' \subseteq S_\tau(\mathcal{T}_1) \times S_\tau(\mathcal{T}_2)$ and $S'' \subseteq S^1 \times S^2$ such that:
  - $s_0 = (s_0^1, s_0^2) \in S'$,
  - If $(s_1, s_2) \in S'$ and there are transitions $s_1 \xrightarrow[\delta_1,\ \lambda_1]{\tau} s_1'$ in $\mathcal{T}_1$ and $s_2 \xrightarrow[\delta_2,\ \lambda_2]{\tau} s_2'$ in $\mathcal{T}_2$ such that $\mu_1(s_1') = \mu_2(s_2')$, then $(s_1', s_2') \in S'$,
  - If $(s_1, s_2) \in S'$ and there are transitions $s_1 \xrightarrow[\delta_1,\ \lambda_1]{\sigma_1} s_1'$ in $\mathcal{T}_1$ and $s_2 \xrightarrow[\delta_2,\ \lambda_2]{\sigma_2} s_2'$ in $\mathcal{T}_2$ for some $\sigma_1, \sigma_2 \in \Sigma$, then $(s_1', s_2')^\star \in S''$.

  Define mappings $\mu_i' : S' \to S_\tau(\mathcal{T}_i)$ as follows: $\mu_i'((s_1, s_2)) = s_i$ $(i = 1, 2)$.
- V is a set of equivalence classes of the equivalence relation R over $V^1 \biguplus V^2$ ($V^1 \biguplus V^2$ is the disjoint union of the sets V^1 and V^2) generated by the relation $R_0 = \{(x_1, x_2) \mid \exists x \in V^0 \diamond \eta_1(x) = x_1 \wedge \eta_2(x) = x_2\}$. Let $\eta_i' : V^i \to V$ $(i = 1, 2)$ send a clock variable from $\mathcal{T}_i$ to the equivalence class to which it belongs.
- $T = T' \cup T''$ such that:

- $((s_1,s_2),\tau,\delta_1[\eta'_1(x)/x] \wedge \delta_2[\eta'_2(x)/x],\eta'_1(\lambda_1) \cup \eta'_2(\lambda_2),(s'_1,s'_2)) \in T' \iff (s_1,s_2),(s'_1,s'_2) \in S'$ and there are the transitions $s_1 \overset{\tau}{\underset{\delta_1,\lambda_1}{\rightarrow}} s'_1$ in $\mathcal{T}_1$ and and $s_2 \overset{\tau}{\underset{\delta_2,\lambda_2}{\rightarrow}} s'_2$ in $\mathcal{T}_2$,
- $((s_1,s_2),a,\delta_1[\eta'_1(x)/x] \wedge \delta_2[\eta'_2(x)/x],\eta'_1(\lambda_1) \cup \eta'_2(\lambda_2),(s'_1,s'_2)^\star) \in T'' \iff (s_1,s_2) \in S'$, $(s'_1,s'_2)^\star \in S''$ and there exist the transitions $s_1 \overset{\sigma'}{\underset{\delta_1,\lambda_1}{\rightarrow}} s'_1$ in $\mathcal{T}_1$ and $s_2 \overset{\sigma''}{\underset{\delta_2,\lambda_2}{\rightarrow}} s'_2$ in $\mathcal{T}_2$ for some $\sigma',\sigma'' \in \Sigma$.

- $I((s_1,s_2)) = I^1(s_1)[\eta'_1(x)/x] \wedge I^2(s_2)[\eta'_2(x)/x]$ for all $(s_1,s_2) \in S$.

Obviously, $\mathcal{T}$ is indeed a timed transition system with invariants. It is routine to show that (μ'_1,η'_1) and (μ'_2,η'_2) are morphisms. We now check that $(\mu_1,\eta_1)\circ(\mu'_1,\eta'_1) = (\mu_2,\eta_2)\circ(\mu'_1,\eta'_2)$. Let $(s_1,s_2) \in S_\tau(\mathcal{T})$. This implies that $(s_1,s_2) \in S'$. By the definition of the set S', we have $\mu_1(s_1) = \mu_2(s_2)$. Then, by the definition of μ'_1 and μ'_2, we know: $\mu_1\circ\mu'_1((s_1,s_2)) = \mu_1(s_1) = \mu_2(s_2) = \mu_2 \circ \mu'_2((s_1,s_2))$. Next, take an arbitrary $x \in V^0$. From the definition of R_0, it follows that $(\eta_1(x),\eta_2(x)) \in R_0$. This means that $\eta_1(x)$ and $\eta_2(x)$ belong to the same class of the equivalence relation, i.e. $\eta'_1(\eta_1(x)) = \eta'_2(\eta_2(x))$. Thus, $\eta'_1 \circ \eta_1 = \eta'_2 \circ \eta_2$.

Suppose $\mathcal{T}_1 \overset{(\phi_1,\xi_1)}{\longleftarrow} \mathcal{T}' \overset{(\phi_2,\xi_2)}{\longrightarrow} \mathcal{T}_2$ to be a construction of morphisms such that $(\mu_1,\eta_1) \circ (\phi_1,\xi_1) = (\mu_2,\eta_2) \circ (\phi_2,\xi_2)$. Define a mapping $\mu : S_\tau(\mathcal{T}') \to S_\tau(\mathcal{T})$ as follows: $\mu(s') = (\phi_1(s'),\phi_2(s'))$. Also, define a mapping $\eta : V \to V'$ as follows: $\eta(x) = \xi_1(x) \cup \xi_2(x)$, where $\xi_i(x) = \{\xi_i(x') \mid x' \in x \wedge x' \in V_i\}$ $(i = 1,2)$. Due to the construction of the set V and the fact that $(\mu_1,\eta_1)\circ(\phi_1,\xi_1) = (\mu_2,\eta_2)\circ(\phi_2,\xi_2)$, it is easy to show that $\xi_1(x)\cup\xi_2(x) = \{z\}$ for some $z \in V'$. It is routine to check that (μ,η) is a morphism. The fact that (μ,η) is a unique morphism such that $(\phi_1,\xi_1) = (\mu'_1,\eta'_1) \circ (\mu,\eta)$ and $(\phi_2,\xi_2) = (\mu'_2,\eta'_2) \circ (\mu,\eta)$ follows from the definition of morphisms. $\diamondsuit$

Following the standards of timed transition systems and [15], we should construct a subcategory of path objects.

Definition 5. Given a timed word $\alpha = (\sigma_1,d_1)\dots(\sigma_n,d_n)$ over Σ_τ, we define a timed transition system with invariants $\mathcal{T}^\alpha = (S^\alpha,0,\Sigma_\tau,V^\alpha,T^\alpha,I^\alpha)$ corresponding to α as follows:

- the set of states, S^α, includes integers in the range from 0 to n, i.e. $S^\alpha = \{0,1,\dots,(n-1),n\}$,
- 0 is the initial state,
- the set of clock variables, V^α, consists of 2^n subsets of states $\{1,2,\dots,n\}$.
- the set of transitions, T^α, for all $i = 0,\dots,n-1$ contains the transition $i \overset{\sigma_i}{\underset{\delta_i,\lambda_i}{\rightarrow}} i+1$, where $\lambda_i = \{x \mid i \in x\}$ and $\delta_i = \bigwedge_{x\in V^\alpha}(x = d_i - d_{I(i,x)})$

with $I(i,x) := \max\{k \in x \cup \{0\} \mid k < i\}$[2] and $d_0 := 0$.

- the invariants I^α is inductively defined as follows: the invariant on the state 0, $I^\alpha(0)$, is $\bigwedge_{x \in V^\alpha} (0 \leq x \leq d_1)$; assume that the invariant on the state $i-1$ is $\bigwedge_{x \in V^\alpha} (c(i-1,x) \leq x \leq \widehat{c}(i-1,x))$, then the invariant on the state i, $I^\alpha(i)$, is $\bigwedge_{x \in V^\alpha}$ (if $i \in x$ then $(0 \leq x \leq (d_{i+1} - d_i))$, else $(\widehat{c}(i-1,x) \leq x \leq \widehat{c}(i-1,x) + (d_{i+1} - d_i)))$, where $d_{n+1} = d_n$.

The class of timed transition systems with invariants of the form $\mathcal{T}^\alpha$ is denoted as **TW**.

Definition 6. The full subcategory $\mathbf{P}_{barbed}$ of the category $\mathcal{TTSI}_{barbed}$ contains timed transition systems with invariants from **TW**, corresponding to timed words over Σ_τ of the forms $(\tau_1, d_1) \ldots (\tau_n, d_n)$ and

$$(\tau_1, d_1) \ldots (\tau_n, d_n)(\sigma, d_{n+1})$$

with $n \geq 0$, $\sigma \in \Sigma$ and $d_i \in \mathbf{R}$ for all $1 \leq i \leq n+1$, and morphisms between them.

Lemma 2. *Let $\mathcal{T}$ be an object of $\mathbf{TTSI}_{barbed}$ and $\mathcal{T}^\alpha$ and $\mathcal{T}^{\alpha(\sigma,d)}$ with $\alpha = (\tau, d_1) \ldots (\tau, d_n)$ and $\sigma \in \Sigma$ be objects of $\mathbf{P}_{barbed}$. Then,*

(i) there is a bijection between the timed words $\beta \in \mathcal{L}(\mathcal{T}^\alpha)$ and the runs of β in $\mathcal{T}^\alpha$. Moreover, the run of α ends in the only final τ-reachable configuration of $\mathcal{T}^\alpha$,

(ii) there is a bijection between the timed words $\beta \in \mathcal{L}(\mathcal{T}^{\alpha(\sigma,d)})$ and the runs of β in $\mathcal{T}^{\alpha(\sigma,d)}$. Moreover, the run of α ends in the only final τ-reachable configuration $\langle s, \nu \rangle$ of $\mathcal{T}^{\alpha(\sigma,d)}$ and $\langle s, \nu \rangle \stackrel{(\sigma,d)}{\rightarrow}$,

(iii) there is a bijection between the morphisms $(\mu, \eta) : \mathcal{T}^\alpha \rightarrow \mathcal{T}$ and the runs of α in $\mathcal{T}$, which are the (μ, η)-images of the run of α in $\mathcal{T}^\alpha$,

(iv) there is a bijection between the morphisms $(\mu, \eta) : \mathcal{T}^{\alpha(\sigma,d)} \rightarrow \mathcal{T}$ and the runs of α in $\mathcal{T}$, which are the (μ, η)-images of the run of α in $\mathcal{T}^{\alpha(\sigma,d)}$ and ends in $\langle s, \nu \rangle$, such that $\langle s, \nu \rangle \stackrel{(\sigma',d)}{\rightarrow}$ for some $\sigma' \in \Sigma$.

Consider the following characterization of $\mathbf{P}_{barbed}$-open maps.

[2] The index returned by $I(i,x)$ is the index of the last state at which x was reset.

Theorem 2. *Let $\mathcal{T} = (S, s_0, \Sigma_\tau, V, T, I)$ and $\mathcal{T}' = (S', \Sigma_\tau, s'_0, V', T', I')$ be timed transition systems with invariants. A morphism $(\mu, \eta) : \mathcal{T} \to \mathcal{T}'$ is $\mathbf{P}_{barbed}$-open iff for any τ-reachable configuration $\langle s, \nu \rangle$ of $\mathcal{T}$ and any clock valuation $\nu' = \nu + d$ such that $\forall d' : d' < d \Rightarrow \|\eta^{-1}(\nu + d')\|_{V'} \in \|I'(\mu(s))\|_{V'}$ the following conditions hold:*

- *whenever there is a transition $\mu(s) \underset{\delta', \lambda'}{\xrightarrow{\tau}} s'_1$ in $\mathcal{T}'$, such that $\|\eta^{-1}(\nu')\|_{V'} \in \|\delta'\|_{V'}$ and $\|\eta^{-1}(\nu')[\lambda' \to 0]\|_{V'} \in \|I'(s'_1)\|_{V'}$, there exists a transition $s \underset{\delta, \lambda}{\xrightarrow{\tau}} s_1$ in $\mathcal{T}$, such that $\mu(s_1) = s'_1$, $\|\nu'\|_V \in \|\delta\|_V$, $\|\nu'[\lambda \to 0]\|_V \in \|I(s_1)\|_V$, $\lambda' = \eta^{-1}(\lambda)$ and $\forall d' : d' < d \Rightarrow \|\nu + d'\|_V \in \|I(s)\|_V$.*
- *whenever there is a transition $\mu(s) \underset{\delta', \lambda'}{\xrightarrow{\sigma}} s'_1$ in $\mathcal{T}'$ for some $\sigma \in \Sigma$, such that $\|\eta^{-1}(\nu')\|_{V'} \in \|\delta'\|_{V'}$ and $\|\eta^{-1}(\nu')[\lambda' \to 0]\|_{V'} \in \|I'(s'_1)\|_{V'}$, there exists a transition $s \underset{\delta, \lambda}{\xrightarrow{\sigma'}} s_1$ in $\mathcal{T}$ for some $\sigma' \in \Sigma$, such that $\|\nu'\|_V \in \|\delta\|_V$, $\|\nu'[\lambda \to 0]\|_V \in \|I(s_1)\|_V$, $\lambda' = \eta^{-1}(\lambda)$ and $\forall d' : d' < d$ $\|\nu + d'\|_V \in \|I(s)\|_V$.*

Proof.

($\Rightarrow$) Assume $(\mu, \eta) : \mathcal{T} \to \mathcal{T}'$ to be a $\mathbf{P}_{barbed}$-open morphism. Take an arbitrary τ-reachable configuration $\langle s, \nu \rangle$ in $\mathcal{T}$ and an arbitrary clock valuation $\nu' = \nu + d$ such that $\forall d' : d' < d \Rightarrow \|\eta^{-1}(\nu + d')\|_{V'} \in \|I'(\mu(s))\|_{V'}$. Suppose that there exists a transition $\mu(s) \underset{\delta', \lambda'}{\xrightarrow{\sigma}} s'_1$ in $\mathcal{T}'$ for some $\sigma \in \Sigma_\tau$, such that $\|\eta^{-1}(\nu')\|_{V'} \in \|\delta'\|_{V'}$ and $\|\eta^{-1}(\nu')[\lambda' \to 0]\|_{V'} \in \|I'(s'_1)\|_{V'}$.

We only consider the case with $\sigma \in \Sigma$ (the case with $\sigma = \tau$ is simpler).

Since $\langle s, \nu \rangle$ is τ-reachable, we have a run γ in $\mathcal{T}$, generating some timed word $\alpha = (\tau, d_1) \ldots (\tau, d_n)$ and ending in $\langle s, \nu \rangle$. From Lemma 2 (iii) for γ we have a unique morphism $(\mu_\gamma, \eta_\gamma) : \mathcal{T}^\alpha \to \mathcal{T}$, defining γ.

Now, let γ' be a (μ, η)-image of γ. Clearly, γ' ends in $\langle \mu(s), \eta^{-1}(\nu) \rangle$. By Lemma 1, γ' is a run in $\mathcal{T}'$, generating the same timed word α. Due to the definition of the relation $\underset{d'}{\xrightarrow{\sigma}}$, we may conclude that $\langle \mu(s), \eta^{-1}(\nu) \rangle$ $\underset{d}{\xrightarrow{\sigma}}$ $\langle s'_1, \eta^{-1}(\nu')[\lambda' \to 0] \rangle$ in $\mathcal{T}'$. From Lemma 2 (iv), it follows that for γ' there exists a unique morphism $(\mu_{\gamma'}, \eta_{\gamma'}) : \mathcal{T}^{\alpha(\sigma,d)} \to \mathcal{T}'$, such that γ' is a $(\mu_{\gamma'}, \eta_{\gamma'})$-image of a run r of α in $\mathcal{T}^{\alpha(\sigma,d)}$. Due to the definition of $\mathcal{T}^\alpha$ and $\mathcal{T}^{\alpha(\sigma,d)}$, we get that there exists only the morphism $(\mu_0, \eta_0) : \mathcal{T}^\alpha \to \mathcal{T}^{\alpha(\sigma,d)}$ in $\mathbf{P}_{barbed}$. Thus, we get a commuting diagram

$$\begin{tikzcd} \mathcal{T}^{\alpha} \arrow[r, "(\mu_\gamma,\, \eta_\gamma)"] \arrow[d, "(\mu_0,\, \eta_0)"'] & \mathcal{T} \arrow[d, "(\mu,\, \eta)"] \\ \mathcal{T}^{\alpha(\sigma,d)} \arrow[r, "(\mu_{\gamma'},\, \eta_{\gamma'})"'] & \mathcal{T}' \end{tikzcd}$$

From the definition of openness, we get a mediating morphism (p, η_p) : $\mathcal{T}^{\alpha(\sigma,d)} \to \mathcal{T}$

$$\begin{tikzcd} \mathcal{T}^{\alpha} \arrow[r, "(\mu_\gamma,\, \eta_\gamma)"] \arrow[d, "(\mu_0,\, \eta_0)"'] & \mathcal{T} \arrow[d, "(\mu,\, \eta)"] \\ \mathcal{T}^{\alpha(\sigma,d)} \arrow[r, "(\mu_{\gamma'},\, \eta_{\gamma'})"'] \arrow[ur, "(p,\eta_p)"'] & \mathcal{T}' \end{tikzcd}$$

such that $(\mu, \eta) \circ (p, \eta_p) = (\mu_{\gamma'}, \eta_{\gamma'})$ and $(p, \eta_p) \circ (\mu_0, \eta_0) = (\mu_\gamma, \eta_\gamma)$. According to Lemma 2 (ii), we have that r is a unique run of α in $\mathcal{T}^{\alpha(\sigma,d)}$ and r ends in $\langle s^r, \nu^r\rangle$ such that $\langle s^r, \nu^r\rangle \underset{d}{\overset{\sigma}{\to}}$. Since $(p, \eta_p) \circ (\mu_0, \eta_0) = (\mu_\gamma, \eta_\gamma)$, we have that γ is a (p, η_p)-image of r and $\langle s, \nu\rangle = \langle p(s^r), \eta_p^{-1}(\nu^r)\rangle$. Hence, from Lemma 1, it follows that $\langle s, \nu\rangle \underset{d}{\overset{\sigma'}{\to}}$ in $\mathcal{T}$ for some $\sigma' \in \Sigma$.

($\Leftarrow$) Suppose that we have a commuting square

$$\begin{tikzcd} \mathcal{T}^{\alpha} \arrow[r, "(\mu_1,\, \eta_1)"] \arrow[d, "(\mu_0,\, \eta_0)"'] & \mathcal{T} \arrow[d, "(\mu,\, \eta)"] \\ \mathcal{T}^{\alpha'} \arrow[r, "(\mu_2,\, \eta_2)"'] & \mathcal{T}' \end{tikzcd}$$

i.e. $(\mu, \eta) \circ (\mu_1, \eta_1) = (\mu_2, \eta_2) \circ (\mu_0, \eta_0)$. W.l.o.g. assume that $\alpha = (\tau, d_1) \ldots (\tau, d_n)$ and $\alpha' = \alpha(\sigma', d')$ $(\sigma' \in \Sigma_\tau)$. Consider the case with $\sigma' = \tau$ (the proof of the case with $\sigma' \in \Sigma$ is similar). Due to Lemma 2 (i), we have a unique run r in $\mathcal{T}^\alpha$, generating α, and a unique run r' in $\mathcal{T}^{\alpha'}$, generating α'. According to Lemma 1, there are runs $\gamma = \langle s_0, \nu_0\rangle \underset{d_1}{\overset{\tau}{\to}} \langle s_1, \nu_1\rangle \ldots \langle s_{n-1}, \nu_{n-1}\rangle \underset{d_n}{\overset{\tau}{\to}} \langle s_n, \nu_n\rangle$ in $\mathcal{T}$, generating α, and $\gamma' = \langle s'_0, \nu'_0\rangle \underset{d_1}{\overset{\tau}{\to}} \langle s'_1, \nu'_1\rangle \ldots \langle s'_{n-1}, \nu'_{n-1}\rangle \underset{d_n}{\overset{\tau}{\to}} \langle s'_n, \nu'_n\rangle \underset{d'}{\overset{\tau}{\to}} \langle s'_{n+1}, \nu'_{n+1}\rangle$ in $\mathcal{T}'$, generating α'. Due to the commutative properties, we have that $\langle \mu(s_i), \eta^{-1}(\nu_i)\rangle = \langle s'_i, \nu'_i\rangle$ for all $0 \le i \le n$. This implies that $\langle \mu(s_n), \eta^{-1}(\nu_n)\rangle \underset{d'}{\overset{\sigma'}{\to}} \langle s'_{n+1}, \nu'_{n+1}\rangle$ in $\mathcal{T}'$. This means that there is a transition $\mu(s_n) \underset{\delta',\lambda'}{\overset{\sigma'}{\to}} s'_{n+1}$ in $\mathcal{T}'$ such that $\|\eta^{-1}(\nu_n + d' - d_n)\|_{V'} \in \|\delta'\|_{V'}$, $\|\eta^{-1}(\nu_n + d' - d_n)[\lambda' \to 0]\|_{V'} \in \|I'(s'_{n+1})\|_{V'}$ and for all $\bar{d}$ such that $0 \le \bar{d} \le d' - d_n$ $\|\eta^{-1}(\nu_n + \bar{d})\|_{V'} \in \|I'(\mu(s_n))\|_{V'}$. Due to the theorem assumptions, we have a transition $s_n \xrightarrow[\delta,\lambda]{\sigma'} s_{n+1}$ such that $\mu(s_{n+1}) = s'_{n+1}$, $\|\nu_n + d' - d_n\|_V \in \|\delta\|_V$, $\|\nu_n + d' - d_n[\lambda \to 0]\|_V \in \|I(s_{n+1})\|_V$ and for all $\bar{d}$ such that $0 \le \bar{d} \le d' - d_n$ the following holds: $\|\nu_n + \bar{d}\|_V \in \|I(s_n)\|_V$. Hence, we get that $\langle s_n, \nu_n\rangle \xrightarrow[d']{\sigma'} \langle s_{n+1}, \nu_{n+1}\rangle$ in $\mathcal{T}$, where $\nu_{n+1} = (\nu_n + d' - d_n)[\lambda \to 0]$, i.e. γ can be extended by $\xrightarrow[d']{\sigma'}$ to run, say, γ_1 in $\mathcal{T}$.

According to Lemma 2 (iii), we have a morphism $(\widetilde{\mu}, \widetilde{\eta}) : \mathcal{T}^{\alpha'} \to \mathcal{T}$, defining γ_1. Moreover, it is easy to see that $(\mu, \eta) \circ (\widetilde{\mu}, \widetilde{\eta}) = (\mu_2, \eta_2)$ and $(\widetilde{\mu}, \widetilde{\eta}) \circ (\mu_0, \eta_0) = (\mu_1, \eta_1)$. $\diamondsuit$

Now, the coincidence of $\mathbf{P}_{barbed}$-bisimilarity and the barbed bisimulation is established.

Theorem 3. *Timed transition systems are $\mathbf{P}_{barbed}$-bisimilar iff they are timed barbed bisimilar.*

Proof.

($\Rightarrow$) Suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ are $\mathbf{P}_{barbed}$-bisimilar. This means that there exists a span of $\mathbf{P}_{barbed}$-open morphisms: $\mathcal{T}_1 \xleftarrow{(\mu_1,\eta_1)} \mathcal{T} \xrightarrow{(\mu_2,\eta_2)} \mathcal{T}_2$. Define $\mathcal{B} \subseteq Conf_\tau(\mathcal{T}_1) \times Conf_\tau(\mathcal{T}_2)$ as follows:

$$(\langle \mu_1(s_i), \eta_1^{-1}(\nu_i)\rangle,\ \langle \mu_2(s_i), \eta_2^{-1}(\nu_i)\rangle) \in \mathcal{B} \text{ for } 0 \le i \le n \overset{def}{\iff}$$
$$\text{there exists a run } \gamma = \langle s_0, \nu_0\rangle \xrightarrow[d_1]{\tau} \langle s_1, \nu_1\rangle \ \ldots\ \langle s_{n-1}, \nu_{n-1}\rangle \xrightarrow[d_n]{\tau} \langle s_n, \nu_n\rangle \text{ in } \mathcal{T}.$$

We have to show that $\mathcal{B}$ is a timed barbed bisimulation. Let $\langle s_0, \nu_0\rangle$ be the initial configuration of $\mathcal{T}$. Since (μ_i, η_i) is a morphism, we have $\langle \mu_i(s_0), \eta_i^{-1}(\nu_0)\rangle = \langle s_0^i, \nu_0^i\rangle$, for all $i = 1, 2$. Thus, it holds that $(\langle s_0^1, \nu_0^1\rangle, \langle s_0^2, \nu_0^2\rangle) \in \mathcal{B}$, by the construction of $\mathcal{B}$.

Take an arbitrary pair $(\langle s^1, \nu^1\rangle, \langle s^2, \nu^2\rangle) \in \mathcal{B}$. W.l.o.g. this implies that there is a run $\gamma = \langle s_0, \nu_0\rangle \xrightarrow[d_1]{\tau} \langle s_1, \nu_1\rangle \ \ldots\ \langle s_{n-1}, \nu_{n-1}\rangle \xrightarrow[d_n]{\tau} \langle s_n, \nu_n\rangle$ in $\mathcal{T}$, such that $\langle \mu_i(s_n), \eta_i^{-1}(\nu_n)\rangle = \langle s^i, \nu^i\rangle$ for all $i = 1, 2$.

We further treat four cases:

– $\langle s^1, \nu^1\rangle \xrightarrow[d]{\tau} \langle s'^1, \nu'^1\rangle$ in $\mathcal{T}_1$. This means that there is a transition $\mu_1(s_n) \xrightarrow[\delta_1,\lambda_1]{\tau} s'^1$ in $\mathcal{T}_1$ such that $\|\eta_1^{-1}(\nu_n + d - d_n)\|_{V_1} \in \|\delta_1\|_{V_1}$, $\|\eta_1^{-1}(\nu_n + d - d_n)[\lambda_1 \to 0]\|_{V_1} \in \|I_1(s'^1)\|_{V_1}$ and for all $\bar{d}$ such that $0 \le \bar{d} \le d - d_n$ $\|\eta_1^{-1}(\nu_n + \bar{d})\|_{V_1} \in \|I_1(\mu_1(s_n))\|_{V_1}$. Due to Theorem 2, we have a transition $s_n \xrightarrow[\delta,\lambda]{\tau} s_{n+1}$ in $\mathcal{T}$ such that $\mu_1(s_{n+1}) = s'^1$, $\lambda_1 = \eta_1^1(\lambda)$, $\|\nu_n + d - d_n\|_V \in \|\delta\|_V$, $\|(\nu_n + d - d_n)[\lambda \to 0]\|_V \in \|I(s_{n+1})\|_V$ and for all $\bar{d}$ such that $0 \le \bar{d} \le d - d_n$ $\|\nu_n + \bar{d}\|_V \in \|I(s_n)\|_V$, since (μ_1, η_1) is a $\mathbf{P}_{barbed}$-open morphism. Hence, $\langle s_n, \nu_n\rangle \xrightarrow[d]{\tau} \langle s_{n+1}, \nu_{n+1}\rangle$

in $\mathcal{T}$, where $\nu_{n+1} = (\nu_n + d - d_n)[\lambda \to 0]$. Moreover, it is easy to check that $\langle \mu_1(s_{n+1}), \eta_1^{-1}(\nu_{n+1})\rangle = \langle s'^1, \nu'^1\rangle$. From the construction of $\mathcal{B}$, it follows that $(\langle s'^1, \nu'^1\rangle, \langle s'^2, \nu'^2\rangle) \in \mathcal{B}$, where $\langle s'^2, \nu'^2\rangle = \langle \mu_2(s_{n+1}), \eta_2^{-1}(\nu_{n+1})\rangle$.

– $\langle s^2, \nu^2\rangle \xrightarrow[d]{\tau} \langle s'^2, \nu'^2\rangle$ in $\mathcal{T}_2$. The proof is symmetric to that of the previous case.

– $\langle s^1, \nu^1\rangle \xrightarrow[d]{\sigma_1}$ in $\mathcal{T}_1$ for some $\sigma_1 \in \Sigma$. This means that there is a transition $\mu_1(s_n) \xrightarrow[\delta_1,\lambda_1]{\sigma_1} s'^1$ in $\mathcal{T}_1$ such that $\|\eta_1^{-1}(\nu_n+d-d_n)\|_{V_1} \in \|\delta_1\|_{V_1}$, $\|\eta_1^{-1}(\nu_n+d-d_n)[\lambda_1 \to 0]\|_{V_1} \in \|I_1(s'^1)\|_{V_1}$ and for all $\bar{d}$ such that $0 \leq \bar{d} \leq d-d_n$ $\|\eta_1^{-1}(\nu_n + \bar{d})\|_{V_1} \in \|I_1(\mu_1(s_n))\|_{V_1}$. Due to Theorem 2, we have a transition $s_n \xrightarrow[\delta,\lambda]{\sigma} s_{n+1}$ in $\mathcal{T}$ for some $\sigma \in \Sigma$ such that $\lambda_1 = \eta_1^1(\lambda)$, $\|\nu_n + d - d_n\|_V \in \|\delta\|_V$, $\|(\nu_n + d - d_n)[\lambda \to 0]\|_V \in \|I(s_{n+1})\|_V$ and for all $\bar{d}$ such that $0 \leq \bar{d} \leq d - d_n$ $\|\nu_n + \bar{d}\|_V \in \|I(s_n)\|_V$, since (μ_1, η_1) is a $\mathbf{P}_{barbed}$-open morphism. Next, due to (μ_2, η_2) being a morphism, we get a transition $\mu_2(s_n) \xrightarrow[\delta_2,\lambda_2]{\sigma_2} s'^2$ in $\mathcal{T}_2$ for some $\sigma_2 \in \Sigma$ such that $\lambda_2 = \eta_2^{-1}(\lambda)$, $\|\eta_2^{-1}(\nu_n+d-d_n)\|_{V_2} \in \|\delta_2\|_{V_2}$, $\|(\eta_2^{-1}(\nu_n+d-d_n))[\lambda_2 \to 0]\|_{V_2} \in \|I_2(s'^2)\|_V$ and for all $\bar{d}$ such that $0 \leq \bar{d} \leq d - d_n$ $\|\eta_2^{-1}(\nu_n) + \bar{d}\|_{V_2} \in \|I_2(\mu_2(s_n))\|_{V_2}$. Hence, $\langle s^2, \nu^2\rangle = \langle \mu_2(s_n), \eta_2^{-1}(\nu_n)\rangle \xrightarrow[d]{\sigma_2} \langle s'^2, \nu'^2\rangle$ in $\mathcal{T}_2$, where $\nu'^2 = \eta_2^{-1}((\nu_n + d - d_n)[\lambda \to 0]) = (\eta 2^{-1}(\nu_n) + d - d_n)[\lambda_2 \to 0]$.

– $\langle s^2, \nu^2\rangle \xrightarrow[d]{\sigma_2}$ in $\mathcal{T}_2$ for some $\sigma_2 \in \Sigma$. The proof is symmetric to that of the previous case.

This means that $\mathcal{B}$ satisfies the required properties of Definition 3.

($\Leftarrow$) Assume that $\mathcal{T}_1$ and $\mathcal{T}_2$ are timed barbed bisimilar. This implies that there is a barbed bisimulation $\mathcal{R} \subseteq Conf_\tau(\mathcal{T}_1) \times Conf_\tau(\mathcal{T}_2)$. We construct a span of $\mathbf{P}_{barbed}$-open maps with a vertex $\mathcal{T} = (S, s_0, \Sigma, V, T, I)$ defined as follows:

– $S = S' \cup S''$, where $S' = \{(\gamma_n^1, \gamma_n^2) \mid \gamma_n^1$ and γ_n^2 are $\mathcal{R}$-related runs$\}$, where

a run $\gamma_n^1 = \langle s_0^1, \nu_0^1\rangle \xrightarrow[d_1]{\tau} \langle s_1^1, \nu_1^1\rangle \ldots \langle s_{n-1}^1, \nu_{n-1}^1\rangle \xrightarrow[d_n]{\tau} \langle s_n^1, \nu_n^1\rangle$ in $\mathcal{T}_1$

and

a run $\gamma_n^2 = \langle s_0^2, \nu_0^2\rangle \xrightarrow[d_1]{\tau} \langle s_1^2, \nu_1^2\rangle \ldots \langle s_{n-1}^2, \nu_{n-1}^2\rangle \xrightarrow[d_n]{\tau} \langle s_n^2, \nu_n^2\rangle$ in $\mathcal{T}_2$,

both generating $\alpha = (\tau, d_1) \ldots (\tau, d_n)$ $(n \geq 0)$, are $\mathcal{R}$-related iff

$$(\langle s_j^1, \nu_j^1\rangle, \langle s_j^2, \nu_j^2\rangle) \in \mathcal{R} \text{ for } 0 \leq j \leq n,$$

$S'' = \{((s^1, \nu^1)_{\gamma_n^1}, (s^2, \nu^2)_{\gamma_n^2}) \mid \gamma_n^1$ and γ_n^2 are $\mathcal{R}$-related runs, ending in $\langle s_n^1, \nu_n^1\rangle$ and $\langle s_n^2, \nu_n^2\rangle$, respectively, and for all $i = 1, 2$ $\langle s_n^i, \nu_n^i\rangle \xrightarrow[d]{\sigma_i}$

$\langle s^i, \nu^i \rangle$ for some σ_i };

- $s_0 = (\langle s_0^1, \nu_0^1 \rangle, \langle s_0^2, \nu_0^2 \rangle)$;
- $V = V_1 \uplus V_2$;
- $T = T' \cup T''$, where
$T' = \{((\gamma_{n-1}^1, \gamma_{n-1}^2) \xrightarrow[\delta_n, \lambda_n]{\tau} (\gamma_n^1, \gamma_n^2)) \mid \gamma_n^1$ and γ_n^2 are runs of the form defined above, γ_{n-1}^1 and γ_{n-1}^2 are runs ending in $\langle s_{n-1}^1, \nu_{n-1}^1 \rangle$ and $\langle s_{n-1}^2, \nu_{n-1}^2 \rangle$, respectively, $\delta_n = \underset{x \in V_1}{\wedge} (x = \nu_{n-1}^1(x) + (d_n - d_{n-1})) \bigwedge \underset{x \in V_2}{\wedge} (x = \nu_{n-1}^2(x) + (d_n - d_{n-1}))$, $\lambda_n = \{x \in V_i \mid i = 1, 2, \nu_n^i(x) = 0\}\}$,
and $T'' = \{((\gamma_n^1, \gamma_n^2) \xrightarrow[\delta, \lambda]{a} ((s^1, \nu^1)_{\gamma_n^1}, (s^2, \nu^2)_{\gamma_n^2})) \mid \gamma_n^1$ and γ_n^2 are runs of the form defined above, $\langle s_n^i, \nu_n^i \rangle \xrightarrow[d]{\sigma^i} \langle s^i, \nu^i \rangle$ in $\mathcal{T}_i$ for all $i = 1, 2$ and some $\sigma^i \in \Sigma$, $\delta = \underset{x \in V_1}{\wedge} (x = \nu_{n-1}^1(x) + (d - d_n)) \bigwedge \underset{x \in V_2}{\wedge} (x = \nu_{n-1}^2(x) + (d - d_n))$ $\lambda_n = \{x \in V_i \mid i = 1, 2, \nu^i(x) = 0\}\}$;

- for all $(\gamma_n^1, \gamma_n^2) \in S'$, we define $I((\gamma_n^1, \gamma_n^2)) = I_1(s_n^1) \bigwedge I_2(s_n^2)$
and for all $((s^1, \nu^1)_{\gamma_n^1}, (s^2, \nu^2)_{\gamma_n^2}) \in S''$, we define

$$I(((s^1, \nu^1)_{\gamma_n^1}, (s^2, \nu^2)_{\gamma_n^2})) = I_1(s^1) \bigwedge I_2(s^2).$$

It is clear that $S_\tau(\mathcal{T}) = S'$.

For $i = 1, 2$, we define mappings $\mu_i : S_\tau(\mathcal{T}) \to S_\tau(\mathcal{T}_i)$ and $\eta_i : V_i \to V$ as follows: $\mu_i((\gamma_n^1, \gamma_n^2)) = s_n^i$ and η_i is the injection function from V_i to $V_1 \uplus V_2$. By the definition of $\mathcal{T}$, (μ_1, η_1) and (μ_2, η_2) are morphisms.

Now, we have to show that (μ_i, η_i) is a $\mathbf{P}_{barbed}$-open morphism $(i = 1, 2)$. Take an arbitrary τ-reachable configuration $\langle (\gamma_1, \gamma_2), \nu \rangle$ in $\mathcal{T}$ and a clock valuation $\nu' = \nu + d$ such that for all $d' < d$ we have $\|\eta_i^{-1}(\nu + d')\|_{V_i} \in \|I_i(\mu_i((\gamma_1, \gamma_2)))\|_{V_i}$. Since (γ_1, γ_2) is a state of $\mathcal{T}$, we get that γ_1 and γ_2 are $\mathcal{R}$-related runs. W.l.o.g., suppose γ_j ends in $\langle s_j, \nu_j \rangle$ for all $j = 1, 2$. It is obvious that $\langle \mu_i(\gamma_1, \gamma_2), \eta_i^{-1}(\nu) \rangle = \langle s_i, \nu_i \rangle$. Assume that $s_i \xrightarrow[\delta_i, \lambda_i]{\sigma} s_i'$ in $\mathcal{T}_i$ for some $\sigma \in \Sigma_\tau$, and that $\|\eta_i^{-1}(\nu')\|_{V_i} \in \|\delta_i\|_{V_i}$ and $\|\eta_i^{-1}(\nu')[\lambda_i \to 0]\|_{V_i} \in \|I_i(s_i')\|_{V_i}$. This implies $\langle s_i, \nu_i \rangle \xrightarrow[d]{\sigma} \langle s_i', \nu_i' \rangle$ in $\mathcal{T}_i$, where $\nu_i' = (\nu_i + d)[\lambda_i \to 0]$. We consider only the case with $\sigma = \tau$ (the proof of the case with $\sigma \in \Sigma$ is similar). Due to $\mathcal{R}$ being a barbed bisimulation, we get that there is a configuration $\langle s_{3-i}', \nu_{3-i}' \rangle$ in $\mathcal{T}_{3-i}$ such that $\langle s_{3-i}, \nu_{3-i} \rangle \xrightarrow[d]{\tau} \langle s_{3-i}', \nu_{3-i}' \rangle$ and $(\langle s_1', \nu_1' \rangle, \langle s_2', \nu_2' \rangle) \in \mathcal{R}$. This implies that, in $\mathcal{T}_j$, the run γ_j can be extended by some τ-timed transition $\langle s_j, \nu_j \rangle \xrightarrow[d]{\tau} \langle s_j', \nu_j' \rangle$ to a run, say, γ_j' for all $j = 1, 2$. Clearly, γ_1' and γ_2' are $\mathcal{R}$-related runs. Furthermore, we may conclude that the exists a transition $s_{3-i} \underset{\delta_{3-i},\lambda_{3-i}}{\overset{\sigma}{\longrightarrow}} s'_{3-i}$ in $\mathcal{T}_{3-i}$, such that $\|\eta_{3-i}^{-1}(\nu')\|_{V_{3-i}} \in \|\delta_{3-i}\|_{V_{3-i}}$ and $\|\eta_{3-i}^{-1}(\nu')[\lambda_{3-i} \to 0]\|_{V_{3-i}} \in \|I_{3-i}(s'_{3-i})\|_{V_{3-i}}$. From the construction of $\mathcal{T}$, it follows that $(\gamma_1, \gamma_2) \underset{\delta,\lambda}{\overset{\tau}{\longrightarrow}} (\gamma'_1, \gamma'_2)$ in $\mathcal{T}$, where $\delta = \bigwedge_{x \in V_1}(x = \nu_1(x) + (d - d_n)) \bigwedge \bigwedge_{x \in V_2}(x = \nu_2(x) + (d - d_n))$, $\lambda = \{x \in V_i \mid i = 1, 2, \nu_i(x) = 0\}$ and $I((\gamma'_i, \gamma'_2)) = I_1(s'_1) \bigwedge I_2(s'_2)$. Moreover, it is easily seen that $\mu_i(\gamma'_1, \gamma'_2) = s'_i$, $\|\nu'\|_V \in \|\delta\|_V$, $\|\nu'[\lambda \to 0]\|_V \in \|I((\gamma'_1, \gamma'_2))\|_V$, $\lambda_i = \eta_i^{-1}(\lambda)$ and for all $d' < d$ it holds $\|\nu + d'\|_V \in \|I((\gamma_1, \gamma_2))\|_V$.

Thus, (μ_i, η_i) is a $\mathbf{P}_{barbed}$-open morphism by Theorem 2. $\diamondsuit$

5. Path-Bisimulation

To obtain a logic characteristic of the bisimulation induced by open maps, Joyal, Nielsen, and Winskel [15] have proposed a second category-theoretic characterization of bisimulation — path bisimulation which is a relation based generalization of the open maps bisimulation.

Definition 7. Let $\mathbb{M}$ be a category of models, $\mathbb{P}$ be a small category of path objects, where $\mathbb{P}$ is a subcategory of $\mathbb{M}$, and I be a common initial object[3] of $\mathbb{P}$ and $\mathbb{M}$. Then,

- Two objects X_1 and X_2 of $\mathbb{M}$ are called *path-$\mathbb{P}$-bisimilar* iff there is a set $\mathcal{R}$ of pairs of paths (p_1, p_2) with a common domain P, so $p_1 : P \to X_1$ is a path in X_1 and $p_2 : P \to X_2$ is a path in X_2, such that

  (o) $(i_1, i_2) \in \mathcal{R}$, where $i_1 : I \to X_1$ and $i_2 : I \to X_2$ are the unique paths starting in the initial object, and for all $(p_1, p_2) \in \mathcal{R}$ and for all $m : P \to Q$, where m is in $\mathbb{P}$, it holds that

  (i) if there exists $q_1 : Q \to X_1$ with $q_1 \circ m = p_1$, then there exists $q_2 : Q \to X_2$ with $q_2 \circ m = p_2$ and $(q_1, q_2) \in \mathcal{R}$ and

  (ii) if there exists $q_2 : Q \to X_2$ with $q_2 \circ m = p_2$, then there exists $q_1 : Q \to X_1$ with $q_1 \circ m = p_1$ and $(q_1, q_2) \in \mathcal{R}$.

- Two objects X_1 and X_2 are *strong path-$\mathbb{P}$-bisimilar* iff they are path-$\mathbb{P}$-bisimilar and the set $\mathcal{R}$ further satisfies:

  (iii) If $(q_1, q_2) \in \mathcal{R}$, with $q_1 : Q \to X_1$ and $q_2 : Q \to X_2$ and $m : P \to Q$, where m is in $\mathbb{P}$, then $(q_1 \circ m, q_2 \circ m) \in \mathcal{R}$.

Theorem 4. *$\mathbf{P}_{barbed}$-bisimilarity, path-$\mathbf{P}_{barbed}$-bisimilarity, and strong path-$\mathbf{P}_{barbed}$-bisimilarity all coincide with the timed barbed bisimilarity.*

[3]In the cases when $\mathbb{P}$ is $\mathbf{P}_{barbed}$ and $\mathbb{M}$ is $\mathcal{TTSI}_{barbed}$, the initial object I_{barbed} is the timed transition system $\mathcal{E} = (\{s_0\}, s_0, \Sigma_\tau, \{x\}, \emptyset, x = 0)$.

Proof. The fact that $\mathbf{P}_{barbed}$-bisimilarity implies (strong) path-$\mathbf{P}_{barbed}$-bisimilarity follows from Lemma 16 [15]. According to Theorem 3, it is sufficient to show that if two timed transition systems are path-$\mathbf{P}_{barbed}$-bisimilar, then they are timed barbed bisimilar.

Assume that $\mathcal{R}$ is a path-$\mathbf{P}_{barbed}$-bisimulation between timed transition systems with invariants $\mathcal{T}_1$ and $\mathcal{T}_2$ over Σ_τ. Define a relation $\mathcal{B} \subseteq Conf_\tau(\mathcal{T}_1) \times Conf_\tau(\mathcal{T}_2)$ as follows:

$$(\langle s_1, \nu_1\rangle, \langle s_2, \nu_2\rangle) \in \mathcal{B} \overset{def}{\Longleftrightarrow} \text{ there is } ((\mu_1, \eta_1), (\mu_2, \eta_2)) \in \mathcal{R} \text{ such that for all } i = 1, 2\ (\mu_i, \eta_i) : P \to \mathcal{T}_i \text{ and } \langle s_i, \nu_i\rangle = \langle \mu_i(s), \eta_i^{-1}(\nu)\rangle, \text{ where } \langle s, \nu\rangle \text{ is the final } \tau\text{-reachable configuration in } P \text{ and } P = \mathcal{T}^\alpha \text{ for some } \alpha = (\tau, d_1) \dots (\tau, d_n).$$

We have to show that $\mathcal{B}$ is a timed barbed bisimilarity. Due to $\mathcal{R}$ being a path-$\mathbf{P}_{barbed}$-bisimilarity, we get $((\mu_1^0, \eta_1^0), (\mu_2^0, \eta_2^0)) \in \mathcal{R}$ such that $(\mu_i^0, \eta_i^0) : I_{barbed} \to \mathcal{T}_i$ for all $i = 1, 2$. Let $\langle s_0, \nu_0\rangle$ be the initial configuration of I_{barbed}. By the construction of I_{barbed}, $\langle s_0, \nu_0\rangle$ is the final τ-reachable configuration of I_{barbed} and $I_{barbed} = \mathcal{T}^\varepsilon$, where ε is an empty timed word. Since (μ_i^0, η_i^0) is a morphism, we have $\langle \mu_i^0(s_0), (\eta_i^0)^{-1}(\nu_0)\rangle = \langle s_0^i, \nu_0^i\rangle$, for all $i = 1, 2$. Thus, it holds that $(\langle s_0^1, \nu_0^1\rangle, \langle s_0^2, \nu_0^2\rangle) \in \mathcal{B}$, by the construction of $\mathcal{B}$.

Take an arbitrary pair $(\langle s^1, \nu^1\rangle, \langle s^2, \nu^2\rangle) \in \mathcal{B}$. Then, there is

$$((\mu_1, \eta_1), (\mu_2, \eta_2)) \in \mathcal{R}$$

such that $(\mu_i, \eta_i) : P \to \mathcal{T}_i$ and $\langle s^i, \nu^i\rangle = \langle \mu_i(s), \eta_i^{-1}(\nu)\rangle$, for all $i = 1, 2$, where $\langle s, \nu\rangle$ is the final τ-reachable configuration in P and $P = \mathcal{T}^\alpha$ for some timed word $\alpha = (\tau, d_1) \dots (\tau, d_n)$. According to Lemma 2 (i), there is the only run r in $\mathcal{T}^\alpha$, generating α. Clearly, r ends in $\langle s, \nu\rangle$. Using Lemma 2 (iii) for (μ_i, η_i), we can find a unique run γ_i in $\mathcal{T}_i$, generating α. Clearly, γ_i is a (μ_i, η_i)-image of r, for all $i = 1, 2$. This implies that γ_i ends in $\langle s^i, \nu^i\rangle$, for all $i = 1, 2$. We further treat four cases:

– $\langle s^1, \nu^1\rangle \xrightarrow[d]{\tau} \langle s^1_{n+1}, \nu^1_{n+1}\rangle$ in $\mathcal{T}_1$. This means that, in $\mathcal{T}_1$, the run γ_1 can be extended by some τ-timed transition $\langle \mu_1(s), \eta_1^{-1}(\nu)\rangle \xrightarrow[d]{\tau} \langle s^1_{n+1}, \nu^1_{n+1}\rangle$ to a run, say, γ_1'. Let $\alpha' = \alpha(\tau, d)$. According to Lemma 2 (iii), for the run γ_1', we can find a unique morphism $(\mu_1', \eta_1') : \mathcal{T}^{\alpha'} \to \mathcal{T}_1$ in $\mathcal{TTSI}_{barbed}$. Using Lemma 2 (i), we may conclude that there is the only run r' in $\mathcal{T}^{\alpha'}$, generating α' and ending in the final τ-reachable configuration, say, $\langle s', \nu'\rangle$ of $\mathcal{T}^{\alpha'}$. By the construction of $\mathcal{T}^\alpha$ and $\mathcal{T}^{\alpha'}$, there is the only morphism $(f, \eta_f) : \mathcal{T}^\alpha \to \mathcal{T}^{\alpha'}$ in $\mathbf{P}_{barbed}$. Thus, we have the commuting diagram

$$\begin{tikzcd}
\mathcal{T}_1 & \mathcal{T}^{\alpha} \arrow[l, "(\mu_1,\eta_1)"'] \arrow[r, "(\mu_2,\eta_2)"] \arrow[d, "(f,\eta_f)"] & \mathcal{T}_2 \\
& \mathcal{T}^{\alpha'} \arrow[ul, "(\mu_1',\eta_1')"] &
\end{tikzcd}$$

Due to $\mathcal{R}$ being a path-$\mathbf{P}_{barbed}$-bisimulation, there exists a morphism $(\mu_2', \eta_2') : \mathcal{T}^{\alpha'} \to \mathcal{T}_2$ such that $((\mu_1', \eta_1'), (\mu_2', \eta_2')) \in \mathcal{R}$ and $(\mu_2, \eta_2) = (\mu_2', \eta_2') \circ (f, \eta_f)$. According to Lemma 2 (iii), for (μ_2', η_2'), we can find a unique run γ_2' in $\mathcal{T}_2$, generating α'. Clearly, γ_i' is a (μ_i', η_i')-image of r', for all $i = 1, 2$. Hence, γ_2' is an extension of γ_2 by the τ-timed transition $\langle \mu_2(s), \eta_2^{-1}(\nu) \rangle \xrightarrow[d]{\tau} \langle \mu_2'(s'), \eta_2'^{-1}(\nu') \rangle$. From the construction of $\mathcal{B}$, it follows that $(\langle s'^1, \nu'^1 \rangle, \langle s'^2, \nu'^2 \rangle) \in \mathcal{B}$, where $\langle s'^2, \nu'^2 \rangle = \langle \mu_2'(s'), \eta_2'^{-1}(\nu') \rangle$.

– $\langle s^2, \nu^2 \rangle \xrightarrow[d]{\tau} \langle s'^2, \nu'^2 \rangle$ in $\mathcal{T}_2$. The proof is symmetric to that of the previous case.

– $\langle s^1, \nu^1 \rangle \xrightarrow[d]{\sigma}$ in $\mathcal{T}_1$ for some $\sigma \in \Sigma$. According to Lemma 2 (iv), for the run γ_1 we can find a unique morphism $(\mu_1', \eta_1') : \mathcal{T}^{\alpha(\sigma,d)} \to \mathcal{T}_1$ in $\mathcal{TTSI}_{barbed}$. By the construction of $\mathcal{T}^{\alpha}$ and $\mathcal{T}^{\alpha(\sigma,d)}$, there is the only morphism $(f, \eta_f) : \mathcal{T}^{\alpha} \to \mathcal{T}^{\alpha(\sigma,d)}$ in $\mathbf{P}_{barbed}$. Thus, we have the commuting diagram

$$\begin{tikzcd}
\mathcal{T}_1 & \mathcal{T}^{\alpha} \arrow[l, "(\mu_1,\eta_1)"'] \arrow[r, "(\mu_2,\eta_2)"] \arrow[d, "(f,\eta_f)"] & \mathcal{T}_2 \\
& \mathcal{T}^{\alpha(\sigma,d)} \arrow[ul, "(\mu_1',\eta_1')"] &
\end{tikzcd}$$

Due to $\mathcal{R}$ being a path-$\mathbf{P}_{barbed}$-bisimulation, there exists a morphism $(\mu_2', \eta_2') : \mathcal{T}^{\alpha(\sigma,d)} \to \mathcal{T}_2$ such that $((\mu_1', \eta_1'), (\mu_2', \eta_2')) \in \mathcal{R}$ and $(\mu_2, \eta_2) = (\mu_2', \eta_2') \circ (f, \eta_f)$. It is obvious that $\langle f(s), \eta_f^{-1}(\nu) \rangle$ is a final τ-reachable configuration of $\mathcal{T}^{\alpha(\sigma,d)}$. Due to the construction of $\mathcal{T}^{\alpha(\sigma,d)}$, we may conclude that $\langle f(s), \eta_f^{-1}(\nu) \rangle \xrightarrow[d]{\sigma}$ in $\mathcal{T}^{\alpha(\sigma,d)}$. According to Lemma 1, we get that $\langle \mu_2'(f(s)), \eta_2'^{-1}(\eta_f^{-1}(\nu)) \rangle \xrightarrow[d]{\sigma'}$ in $\mathcal{T}_2$ for some $\sigma' \in \Sigma$. Since $(\mu_2, \eta_2) = (\mu_2', \eta_2') \circ (f, \eta_f)$ we have

$$\langle \mu_2'(f(s)), \eta_2'^{-1}(\eta_f^{-1}(\nu)) \rangle = \langle \mu_2(s), \eta_2^{-1}(\nu) \rangle = \langle s^2, \nu^2 \rangle.$$

– $\langle s^2, \nu^2 \rangle \xrightarrow[d]{\sigma}$ in $\mathcal{T}_2$ for some $\sigma \in \Sigma$. The proof is symmetric to that of the previous case.

$\diamondsuit$

6. Decidability

In this section, we consider the decidability questions for $\mathbf{P}_{barbed}$-openness of a morphism and for the timed barbed bisimilarity in the setting of finite timed transition systems with invariants, i.e. systems with a finite set of states and for which all constants referred to in clock constraints and in invariants are natural valued. The subclass of finite timed transition systems with invariants is denoted by $\mathcal{TTSI}_{\mathbf{N}}$.

As for many existing results for timed models, including the results concerning verification of real-time systems, our decision procedure relies heavily on the idea behind regions [1] which essentially provide a finite description of the state-space of timed transition systems with invariants.

Given a finite set of clock variables V and an integer constant c, a region is an equivalence class of the equivalence relation $\simeq$ over clock valuations, where $\nu \simeq \nu'$ iff

- for each $x \in V$, $\lfloor \nu(x) \rfloor = \lfloor \nu'(x) \rfloor$ or both $\nu(x) > c$ and $\nu(x) > c$,
- for each pair $x, y \in V$ such that both $\nu(x) \leq c$ and $\nu(y) \leq c$, $fract(\nu(x)) \leq fract(\nu(y))$ iff $fract(\nu'(x)) \leq fract(\nu'(y))$,
- for each $x \in V$ such that $\nu(x) \leq c$, $fract(\nu(x)) = 0$ iff $fract(\nu'(x)) = 0$.

For a clock valuation ν, let $[\nu]$ denote the region to which it belongs. Let $\mathcal{R}_{V,c}$ denote the (finite) set of regions associated with V and c. Given the regions $reg, reg' \in \mathcal{R}_{V,c}$, $reg' \in Reach(reg)$ iff there exists $\nu \in reg$ and $d \in \mathbf{R}$ such that $\nu + d \in reg'$. Finally, for a finite timed transition system with invariants $\mathcal{T}$, an extended configuration is defined as any pair $\langle s, reg \rangle$, where $s \in S$ and $reg \in \mathcal{R}_{V,c}$. An extended configuration $\langle s, reg \rangle$ is called *τ-reachable* if $\langle s, \nu \rangle$ is a τ-reachable configuration for some $\nu \in reg$.

We can now give a characterization of $\mathbf{P}_{barbed}$-open maps in terms of extended configurations.

Theorem 5. *Consider finite timed transition systems with invariants $\mathcal{T} = (S, s_0, \Sigma_\tau, V, T, I)$ and $\mathcal{T}' = (S', \Sigma_\tau, s_0', V', T', I')$ and associated regions defined with respect to some integer constant greater than or equal to the largest constant referred to in transition constraint expressions and in invariants in $\mathcal{T}$ and $\mathcal{T}'$. A morphism $(\mu, \eta) : \mathcal{T} \to \mathcal{T}'$ is $\mathbf{P}_{barbed}$-open iff for any τ-reachable extended configuration $\langle s, reg \rangle$ of $\mathcal{T}$ and any $reg' \in Reach(reg)$, such that $\forall reg'' \in Reach(reg)$ if $reg' \in Reach(reg'')$ then $\|\eta^{-1}(reg'')\|_{V'} \subseteq \|I'(\mu(s))\|_{V'}$, the following conditions hold:*

- *whenever there is a transition $\mu(s) \xrightarrow[\delta', \lambda']{\tau} s_1'$ in $\mathcal{T}'$, such that*

$\|\eta^{-1}(reg')\|_{V'} \subseteq \|\delta'\|_{V'}$ *and* $\|\eta^{-1}(reg')[\lambda' \to 0]\|_{V'} \subseteq \|I'(s'_1)\|_{V'}$,

there exists a transition $s \xrightarrow[\delta,\lambda]{\tau} s_1$ *in* $\mathcal{T}$ *such that* $\mu(s_1) = s'_1$, $\|reg'\|_V \subseteq \|\delta\|_V$, $\lambda' = \eta^{-1}(\lambda)$, $\|reg'[\lambda \to 0]\|_V \subseteq \|I(s_1)\|_V$ *and* $\forall reg'' \in Reach(reg)$ *if* $reg' \in Reach(reg'') \Rightarrow \|reg''\|_V \subseteq \|I(s)\|_V$.

- *whenever there is a transition* $\mu(s) \xrightarrow[\delta',\lambda']{\sigma'} s'_1$ *in* $\mathcal{T}'$ *for some* $\sigma' \in \Sigma$ *such that* $\|\eta^{-1}(reg')\|_{V'} \subseteq \|\delta'\|_{V'}$ *and* $\|\eta^{-1}(reg')[\lambda' \to 0]\|_{V'} \subseteq \|I'(s'_1)\|_{V'}$, *there exists a transition* $s \xrightarrow[\delta,\lambda]{\sigma} s_1$ *in* $\mathcal{T}$ *for some* $\sigma \in \Sigma$ *such that* $\|reg'\|_V \subseteq \|\delta\|_V$, $\lambda' = \eta^{-1}(\lambda)$, $\|reg'[\lambda \to 0]\|_V \subseteq \|I(s_1)\|_V$ *and* $\forall reg'' \in Reach(reg)$ *if* $reg' \in Reach(reg'') \Rightarrow \|reg''\|_V \subseteq \|I(s)\|_V$.

Proof. Follows from Theorem 2, Proposition 3 of [13] and the property that if one clock evaluation in a region satisfies an invariant, then all the clock evaluations of that region satisfy the invariant. $\diamondsuit$

Notice that Theorem 5 immediately implies the following decidability result of $\mathbf{P}_{barbed}$-openness of a morphism between two finite timed transition systems with invariants, because the number of regions over the set of clocks V with a constant c, is $|V|\,! \, * 2^{|V|} * (2c+2)^{|V|}$.

Corollary 1. *Given* $\mathcal{T}$, $\mathcal{T}' \in \mathcal{TTSI}_{\mathbf{N}}$ *and a morphism* $(\mu,\eta) : \mathcal{T} \to \mathcal{T}'$, $\mathbf{P}_{barbed}$*-openness of* (μ,η) *is decidable.*

In order to establish decidability of the timed barbed bisimulation, we need to prove the following fact.

Theorem 6. *Given two timed transition systems* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *from* $\mathcal{TTSI}_{\mathbf{N}}$, *if there exists a span of* $\mathbf{P}_{barbed}$*-open morphism of the form* $\mathcal{T}_1 \overset{(\mu_1,\eta_1)}{\longleftarrow} \mathcal{T}_0 \overset{(\mu_2,\eta_2)}{\longrightarrow} \mathcal{T}_2$, *then there exists a span of* $\mathbf{P}_{barbed}$*-open morphisms of the form* $\mathcal{T}_1 \overset{(\mu'_1,\eta'_1)}{\longleftarrow} \mathcal{T} \overset{(\mu'_2,\eta'_2)}{\longrightarrow} \mathcal{T}_2$, *where* $\mathcal{T}$ *is a timed transition system with invariants from* $\mathcal{TTSI}_{\mathbf{N}}$ *and its size is bounded by the size of* $\mathcal{T}_1$ *and* $\mathcal{T}_2$.

Proof. Assume that $\mathcal{T}_1$ and $\mathcal{T}_2$ are timed transition systems with invariants from $\mathcal{TTSI}_{\mathbf{N}}$ and $\mathcal{T}_1 \overset{(\mu_1,\eta_1)}{\leftarrow} \mathcal{T}_0 \overset{(\mu_2,\eta_2)}{\rightarrow} \mathcal{T}_2$ is a span of $\mathbf{P}_{barbed}$-open morphisms.

Construct a timed transition system with invariants $\mathcal{T} = (S, s_0, \Sigma_\tau, V, T, I)$ and two morphisms $(\mu'_i, \eta'_i) : \mathcal{T} \to \mathcal{T}_i$ $(i = 1, 2)$ as follows:

- $S = S' \cup S''$, where $S' = \{(s^1, s^2) \in S_\tau(\mathcal{T}_1) \times S_\tau(\mathcal{T}_2) \mid \exists s \in S_\tau(\mathcal{T}_0) \diamond \mu_1(s) = s^1,\ \mu_2(s) = s^2\}$, $S'' = \{s_{(s^1,\delta_1,\lambda_1,I^1,s^2,\delta_2,\lambda_2,I^2)} \mid (s^1, s^2) \in S',\ s^1 \xrightarrow[\delta_1,\ \lambda_1]{\sigma'} s'^1$ in $\mathcal{T}_1$,

$I_1(s'^1) = I^1$, $s^2 \underset{\delta_2,\ \lambda_2}{\xrightarrow{\sigma''}} s'^2$ in $\mathcal{T}_2$, $I_2(s'^2) = I^2$ and $\sigma', \sigma'' \in \Sigma\}$. Define the mappings $\mu_i' : \mathcal{T} \to \mathcal{T}_i$ as $\mu_i'((s^1, s^2)) = s^i$ $(i = 1, 2)$.

- $s_0 = (s_0^1, s_0^2)$,
- $V = V_1 \biguplus V_2$, where $V_1 \biguplus V_2$ is the disjoint union of the sets V_1 and V_2. Define the mappings $\eta_i' : V_i \to V$ $(i = 1, 2)$ as injections,
- $T = T' \cup T''$, where $((s^1, s^2), \tau, \delta_1[\eta_1'(x)/x] \wedge \delta_2[\eta_2'(x)/x], \eta_1'(\lambda_1) \cup \eta_2'(\lambda_2), (s'^1, s'^2)) \in T' \iff (s^1, s^2), (s'^1, s'^2) \in S'$, $s^1 \underset{\delta_1,\ \lambda_1}{\xrightarrow{\tau}} s'^1$ and $s^2 \underset{\delta_2,\ \lambda_2}{\xrightarrow{\tau}} s'^2$, and $((s^1, s^2), \sigma, \delta_1[\eta_1'(x)/x] \wedge \delta_2[\eta_2'(x)/x], \eta_1'(\lambda_1) \cup \eta_2'(\lambda_2), s_{(s^1,\delta_1,\lambda_1,I^1,s^2,\delta_2,\lambda_2,I^2)}) \in T'' \iff (s^1, s^2) \in S'$, $s_{(s^1,\delta_1,\lambda_1,I^1,s^2,\delta_2,\lambda_2,I^2)} \in S''$;
- $I((s^1, s^2)) = I_1(s^1)[\eta_1'(x)/x] \bigwedge I_2(s^2)[\eta_2'(x)/x]$ for all $(s^1, s^2) \in S'$ and $I(s_{(s^1,\delta_1,\lambda_1,I^1,s^2,\delta_2,\lambda_2,I^2)}) = I^1[\eta_1'(x)/x] \bigwedge I^2[\eta_2'(x)/x]$ for all
$$s_{(s^1,\delta_1,\lambda_1,I^1,s^2,\delta_2,\lambda_2,I^2)} \in S''.$$

Clearly, $\mathcal{T}$ is a timed transition system with invariants from $\mathcal{TTSI}_{\mathbf{N}}$, moreover, the size of $\mathcal{T}$ is bounded by the size of $\mathcal{T}_1$ and $\mathcal{T}_2$, because $|S'| \leq |S_1| * |S_2|$, $|S''| \leq |S_1| * |S_2| * |T_1| * |T_2| * |S_1| * |S_2|$, $|V| = |V_1| + |V_2|$ and $|T| \leq |T_1| * |T_2|$.

Next, we shall check that (μ_i', η_i') $(i = 1, 2)$ is a morphism. To verify this fact, we consider the conditions of Definition 4.

- Obviously, we have $\mu_i'((s_0^1, s_0^2)) = s_0^i$.
- Let $(s^1, s^2) \in S_\tau(\mathcal{T})$. Then we have
$$I((s^1, s^2)) = I_1(s^1)[\eta_1'(x)/x] \bigwedge I_2(s^2)[\eta_2'(x)/x].$$
This implies $\|I((s^1, s^2))\|_V \subseteq \|I_i(s^i)[\eta_i'(x)/x]\|_V = \|I_i(\mu_i'(s^i))[\eta_i'(x)/x]\|_V$.
- Assume that $(s_1, s_2) \underset{\delta,\ \lambda}{\xrightarrow{\tau}} (s_1', s_2')$ in $\mathcal{T}$. From the construction of $\mathcal{T}$, it follows that $s_1 \underset{\delta_1,\ \lambda_1}{\xrightarrow{\tau}} s_1'$, $s_2 \underset{\delta_2,\ \lambda_2}{\xrightarrow{\tau}} s_2'$, $\delta = \delta_1[\eta_1'(x)/x] \wedge \delta_2[\eta_2'(x)/x]$, and $\lambda = \eta_1'(\lambda_1) \cup \eta_2'(\lambda_2)$. Clearly, $\mu_i'((s_1, s_2)) = s_i$ and $\mu_i'((s_1', s_2')) = s_i'$. Hence, we have $\mu_i'((s_1, s_2)) \underset{\delta_i,\lambda_i}{\xrightarrow{\tau}} \mu_i'(s_1', s_2')$ in $\mathcal{T}_i$, and, moreover:

    1. $(\eta_i')^{-1}(\lambda) = \{x_i \in V_i \mid \eta_i'(x_i) \in \lambda\} = \{x_i \in V_i \mid \eta_i'(x_i) \in \eta_1'(\lambda_1) \cup \eta_2'(\lambda_2)\} = \{x_i \in V_i \mid \eta_i'(x_i) \in \eta_i'(\lambda_i)\} = \lambda_i$,
    2. $\|\delta\|_V = \|\delta_1[\eta_1'(x)/x] \wedge \delta_2[\eta_2'(x)/x]\|_V \subseteq \|\delta_i[\eta_i'(x)/x]\|_V$.

- Assume that $(s_1, s_2) \underset{\delta,\ \lambda}{\xrightarrow{\sigma}} s*$. From the construction of $\mathcal{T}$, it follows that $s* = s_{(s^1,\delta_1,\lambda_1,I^1,s^2,\delta_2,\lambda_2,I^2)}$ and there exist transitions $(s_1) \underset{\delta_1,\ \lambda_1}{\xrightarrow{\sigma_1}} s'_1$ in $\mathcal{T}_1$ and $(s_2) \underset{\delta_2,\ \lambda_2}{\xrightarrow{\sigma_2}} s'_2$ in $\mathcal{T}_2$ for some $\sigma_1, \sigma_2 \in \Sigma$, such that $\delta = \delta_1[\eta'_1(x)/x] \wedge \delta_2[\eta'_2(x)/x]$, $\lambda = \eta'_1(\lambda_1) \cup \eta'_2(\lambda_2)$, $I^1 = I_1(s'_1)$ and $I^2 = I_2(s'_2)$. Since $\mu'_1((s_1, s_2)) = s_1$, we have $\mu'_1((s_1, s_2) \underset{\delta_i,\ \lambda_i}{\xrightarrow{\sigma_i}} s'_i$ in $\mathcal{T}_i$. Moreover, we get the following:
    1. $(\eta'_i)^{-1}(\lambda) = \lambda_i$,
    2. $\|\delta\|_V = \|\delta_1[\eta'_1(x)/x] \wedge \delta_2[\eta'_2(x)/x]\|_V \subseteq \|\delta_i[\eta'_i(x)/x]\|_V$,
    3. $\|I(s*)\|_V = \|I^1[\eta'_1(x)/x] \bigwedge I^2[\eta'_2(x)/x]\|_V \subseteq \|I^i[\eta'_i(x)/x]\|_V = \|I_i(s'_i)[\eta'_i(x)/x]\|_V$.

Thus, (μ'_i, η'_i) $(i = 1, 2)$ is a morphism.

Finally, we shall show that (μ'_i, η'_i) $(i = 1, 2)$ is a $\mathbf{P}_{barbed}$-open morphism. Take an arbitrary $\langle (s_1, s_2), \nu \rangle \in Conf_\tau(\mathcal{T})$ and $\nu' = \nu + d$ such that $\forall d' : d' < d \Rightarrow \|\eta_i'^{-1}(\nu + d')\|_{V_i} \in \|I_i(\mu'_i((s_1, s_2)))\|_{V_i}$. Due to the definition of $\mathcal{T}$, there exists $s \in S_\tau(\mathcal{T}_0)$ such that $\mu_j(s) = s^j$ for all $j = 1, 2$. It is easy to check that $\langle s, \overline{\nu} \rangle$ is a τ-reachable configuration in $\mathcal{T}_0$, where $\eta_i^{-1}(\overline{\nu}) = \eta_j'^{-1}(\nu)$ for all $j = 1, 2$.

Assume that $\mu'_i((s_1, s_2)) = s_i \underset{\delta_i,\lambda_i}{\xrightarrow{\sigma_i}} s'_i$ is a transition in $\mathcal{T}_i$ such that

$$\|\eta_1'^{-1}(\nu')[\lambda_i \to 0]\|_{V_i} \in \|I_i(s'_i)\|_{V_i}$$

and $\|{\eta'}_i^{-1}(\nu')\|_{V_i} \in \|\delta_i\|_{V_i}$. We consider only the case with $\sigma_i = \tau$ (the proof of the case with $\sigma_i \in \Sigma$ is similar). Since (μ_i, η_i) is a $\mathbf{P}_{barbed}$-open morphism, from Theorem 2 it follows that there exists a transition $s \underset{\delta_0,\lambda_0}{\xrightarrow{\tau}} s'$ in $\mathcal{T}_0$ such that $\mu_i(s') = s'_i$, $\|\overline{\nu} + d\|_{V_0} \in \|\delta_0\|_{V_0}$, $\|(\overline{\nu} + d)[\lambda_0 \to 0]\|_{V_0} \in \|I_0(s')\|_{V_0}$, $\lambda_0 = \eta_i^{-1}(\lambda_i)$ and, for all d' such that $d' < d$, it holds that $\|\overline{\nu} + d'\|_{V_0} \in \|I_0(s)\|_{V_0}$.

Next, we may conclude that there is a transition $\mu_{3-i}(s) \underset{\delta_{3-i},\lambda_{3-i}}{\xrightarrow{\tau}} \mu_{3-i}(s')$ in $\mathcal{T}_{3-i}$ such that
$\mu_{3-i}(s) = s_{3-i}$, $\mu_{3-i}(s') = s'_{3-i}$, $\|\delta_0\|_{V_0} \subseteq \|\delta_{3-i}[\eta_{3-i}^{-1}(x)/x]\|_{V_0}$,
$\lambda_{3-i} = \eta_{3-i}^{-1}(\lambda_0)$ and $\|I_0(s')\|_{V_0} \subseteq \|I_{3-i}(\mu_{3-i}(s))[\eta_{3-i}(x)/x]\|_{V_0}$,
because (μ_{3-i}, η_{3-i}) is a morphism. This means that $\|(\eta_{3-i}'^{-1}(\nu')[\lambda_{3-i} \to 0]\|_{V_{3-i}} \in \|I_{3-i}(s'_{3-i})\|_{V_{3-i}}$ and $\|{\eta'}_{3-i}^{-1}(\nu')\|_{V_{3-i}} \in \|\delta_{3-i}\|_{V_{3-i}}$.

Due to the construction of $\mathcal{T}$, we get a transition $(s_1, s_2) \underset{\delta,\lambda}{\xrightarrow{\tau}} (s'_1, s'_2)$ in $\mathcal{T}$, where $\delta = \delta_1[\eta'_1(x)/x] \wedge \delta_2[\eta'_2(x)/x]$ and $\lambda = \eta'_1(\lambda_1) \cup \eta'_2(\lambda_2)$. It is not difficult to show that $\mu'_i((s'_1, s'_2)) = s'_i$, $\|\nu'\|_V \in \|\delta\|_V$, $\|\nu'[\lambda \to 0]\|_V \in \|I((s'_1, s'_2))\|_V$, $\lambda_i = \eta_i'^{-1}(\lambda)$ and, for all d' such that $d' < d$, it holds that $\|\nu + d'\|_V \in \|I((s_1, s_2))\|_V$.

Thus, by Theorem 2, (μ_i', η_i') is a $\mathbf{P}_{barbed}$-open morphism. $\diamondsuit$

Corollary 2. *Given two finite timed transition systems with invariants $\mathcal{T}$ and $\mathcal{T}'$ from $\mathcal{TTSI}_{\mathbf{N}}$, the timed barbed bisimulation is decidable.*

Proof. Follows from Theorem 6 and Corollary 1. $\diamondsuit$

7. Conclusion

In this paper, we have generalized the category-theoretic approaches introduced in the paper [15] to the timed barbed bisimulation on timed transition systems with invariants illustrating that the bisimulation can also be captured by the idea of span of open maps and by path bisimilarity. This allows us to transfer general concepts of equivalences to the model under consideration and to apply general results from the categorical setting (e.g. existence of canonical models and characteristic games and logics) to concrete time-sensitive equivalences. It is also shown that the decidability result concerning timed bisimulation from [13] can be directly adopted for the timed barbed bisimulation.

In the future, we plan to extend the obtained results to other classes of timed models (e.g. time Petri nets, networks of timed automata, etc.). In particular, relying on the paper [15], we contemplate to adapt the unfolding methods for time Petri nets from [5] and open maps based characterizations for timed event structures from [23] to transfer the general concept of bisimulation to the timed models mentioned above.

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