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$\mathcal{ALC}$ for CLA: Towards description logic on concept lattices*

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Abstract. In this paper, we start with a motivation of the study of modal and/or description logics with values in concept lattices. Then we give a brief survey of approaches to lattice-valued modal and/or description logics. After that we study some methods of context symmetrization, because in our approach the description logic on concept lattices is defined for symmetric contexts only. We conclude with a list of problems related to the comparison of different lattice-valued modal and/or description logics, different variants of context symmetrization and resulting description logics, decidability and axiomatization of these logics.

Keywords: description logic, lattice-valued modal logics, modal logic, distributive lattice, concept lattice.

1. Introduction

Let us start with an example that can explain our interest in the study of polymodal and/or description logics with values in concept lattices.

Let us fix a moment of time and let

- URL be the set of all Uniform Resource Locators that are valid (exist) at this moment,
- Key be the set of all Key-words in any existing language that are conceivable in this time,
- F, S and T be binary relations on $URL \times Key$ that are implemented in some (non-real we assume) search engines First, Second and Third at the moment of time fixed above.

Then let $Sh + Ga$ be the set of all web-sites (represented by their URL's) that a search engine First finds by two key-words *Shilov* and *Garanina*. In terms of Formal Concept Analysis (FCA) [5, 15] $Sh + Ga = \{Shilov, Garanina\}'$ in the formal context $\mathbb{F} = (URL,\ Key,\ F)$.

Similarly, let Gr be the set of all web-sites that Second finds by searching for a single key-word *Grebeneva*. In FCA terms one can write $Gr = \{Grebeneva\}'$ in the formal context $\mathbb{S} = (URL,\ Key,\ S)$.

Assume that we would like to know the set $(Sh + Ga) - Gr$ consisting of

*$\mathcal{ALC}$, *Attribute Language with Complements*, one of the most simple description logic; **CLA**, *Concept Lattices and Their Applications*, the International conference series.

all sites that are found by First
for key-words *Shilov* and *Garanina*
but which (according to Third) do not contain any word
common for all sites found by Second
for the key-word *Grebeneva*.

In terms of set theory expanded by FCA-derivatives, the desired set can be written as $(Sh + Ga) \setminus Gr''$, where prime '$'$' represents a derivative in the formal context $\mathbb{T} = (URL,\ Key,\ Third)$.

Recall that $Sh + Ga = \{Shilov, Garanina\}'$ in $\mathbb{F}$, $Gr = \{Grebeneva\}'$ in $\mathbb{S}$. Since we use three different contexts $\mathbb{F}$, $\mathbb{S}$ and $\mathbb{T}$, it is not correct to write

$$(Sh + Ga) - Gr = \{Shilov, Garanina\}' \setminus \{Grebeneva\}''',$$

but we have to write

$$(Sh + Ga) - Gr = \{Shilov, Garanina\}^{\downarrow_F} \setminus \{Grebeneva\}^{\downarrow_S\uparrow_T\downarrow_T},$$

where $\downarrow_F$ represents the lower derivative in the context $\mathbb{F}$, $\downarrow_S$ — the lower derivative in the context $\mathbb{S}$, and $\downarrow_T$ and $\uparrow_T$ — the lower and upper derivatives in the context $\mathbb{T}$.

We believe that queries similar to the above are quite natural, meaningful and useful. However, processing and solving such complicated queries is above the abilities of modern search engines. This inability is partially attributed to the lack of formal semantics of such multi-context queries.

At the same time, polymodal and/or descriptive logics (DL) [1, 2] provide a language for presentation of queries as above. In particular, if we denote by T^d the inverse of the binary relation T, then $(Sh + Ga) - Gr$ may be represented in the syntax of a polymodal logic by the following formula

$$[F](Shilov\&Garanina)\ \&\ \neg[T][T^d][S]Grebeneva$$

or in the syntax of a description logic as the following concept term

$$\forall F.(Shilov \sqcap Garanina)\ \sqcap\ \neg\forall T.\forall T^d.\forall S.Grebeneva.$$

The interpretation of FCA constructs in DL was studied in [11, 13]. In these studies, DL was extended by FCA-derivatives and provided with Kripke semantics, concept terms were interpreted by sets of objects, but not as elements of a concept lattice.

A variant of a negation-free description logic (namely, the negation-free $\mathcal{ALC}$, the Attribute Language with Complements) with values in concept lattices was defined in [12]. It turns out that this lattice-valued semantics can be extended for complete $\mathcal{ALC}$ in *symmetric* contexts (i.e. in contexts where sets of objects and attributes are equal and the binary relation is symmetric). This implies that if we wish to extend the lattice semantics defined in [12] to more expressive description logics than the positive fragment of $\mathcal{ALC}$, then we have to study ways to *symmetrise* contexts, i.e. how to build a symmetric context from a given one and how to embed this given context into it (the built symmetric context). In this paper, we present some preliminary results of our studies into the ways of context symmetrisation, formulate and discuss some topics needing more research.

2. Lattice-valued modal and description logics

Modal and Description Logic are closely related but have different research paradigms: they have different syntax and pragmatic, but very closely related semantics (in spite of different terminology).

2.1. Lattice-valued modal logics

Lattice-valued modal logics were introduced in [4, 3] by M.C. Fitting. They were studied in the cited papers from a proof-theoretic point of view. Later several authors attempted a study of these logics from the algebraic perspective [7, 8, 14]. Basic definitions related to modal logics on lattices follow below.

Let L be a fixed finite distributive lattice, where $\bot$ and $\top$ denote the smallest and the greatest elements, and $\to_L$ denotes the relative pseudo-complement (where the subscript L may be omitted when L is implicit).

Definition 1.
For for all $a \in L$ let $\lambda x \in L.T_a(x)$ be a unary operation on L such that for all $b \in L$ (1) $T_a(b) = \top$ (if $b = a$) and (2) $T_a(b) = \bot$ (if $b \neq a$).

Definition 2.

- The set of formulas **Form** of the L-valued logic (L-**VL**) is defined by induction in a usual way by using the set of propositional variables **PV**, the logical connectives $\wedge$, $\vee$, $\to$, 0, 1 and T_a for each $a \in L$.
- L-valuation is any total function v from **Form** to L that satisfies the following properties:
  - $v(T_a(x)) = T_a(v(x))$;
  - $v(x \wedge y) = inf(v(x), v(y))$;
  - $v(x \vee y) = sup(v(x), v(y))$;
  - $v(x \to y) = v(x) \to_L v(y)$;
  - $v(0) = \bot$ and $v(1) = \top$.

Then, a formula x is said to be a valid formula of L-**VL** iff $v(x) = \top$ for all L-valuations v. If L is the two-element Boolean algebra, the valid formulas of L-**VL** coincide with the tautologies of the classical propositional logic. Since L is finite, the L-**VL** validity problem is decidable.

Definition 3.

- The syntax of L-valued modal logic L-**ML** extends the syntax of L-**VL** by modality $\Box$. $\mathbf{Form}_\Box$ denotes the set of formulas of L-**ML**.
- Let (M, R) be a Kripke frame. Then, v is a Kripke L-valuation on (M, R) if v is a function from $M \times \mathbf{Form}_\Box$ to L that satisfies the following properties for each $w \in M$:
  - $v(w, \Box x) = \bigwedge_{wRw'} v(w', x)$;
  - $v(w, T_a(x)) = T_a(v(w, x))$;
  - $v(w, x \wedge y) = inf(v(w, x), v(w, y))$;
  - $v(w, x \vee y) = sup(v(w, x), v(w, y))$;
  - $v(w, x \to y) = v(w, x) \to_L v(w, y)$;
  - $v(0) = \bot$ and $v(1) = \top$.

  Then (M, R, v) is called L-valued (modal) Kripke model.

Again, a formula x is said to be a valid L-**ML** formula if $v(w, x) = \top$ for every L-valued Kripke model (M, R, v) and every $w \in M$. Completeness and the finite model property for L-**ML** are proved in [7].

If L is the two-element Boolean algebra, the valid formulas of L-**ML** coincide with those of the classical modal logic **K**.

If we consider the class of Kripke frames with a reflexive and transitive relation R, then we get L-valued **S4**-type modal logic L-**S4**. For this logic the completeness theorem is proved also in [7].

2.2. $\mathcal{ALC}$ with values in concept lattices

Description Logic (DL) is a logic for reasoning about concepts. Also there is an algebraic formalism developed around concepts in terms of concept lattices, namely Formal Concept Analysis (FCA). In this section we recall in brief the definition of the description logic $\mathcal{ALC}$ on concept lattices of (symmetric) contexts and some properties that follow from this definition[1]. We use notation and definitions for Description Logics from [1][2]. For the basics and notation of Formal Concept Analysis, please refer to [5].

[1]Please refer to [12] for full details.

[2]For the sake of readability, we use Υ instead of '$\cdot$' for the terminological interpretation function.

Semantics of description logics on concept lattices comes from lattice-theoretic characterization of *positive* (i.e. without negation) concept constructs (for close world semantics) given in the following proposition [12].

Proposition 1. *Let* (Δ, Υ) *be a terminological interpretation and* $P(\Delta) = (2^{\Delta}, \varnothing, \subseteq, \Delta, \cup, \cap)$ *be the complete lattice of subsets of* Δ. *Then semantics of the* $\mathcal{ALC}$ *positive concept constructs* $\top$, $\bot$, $\sqcup$, $\sqcap$, $\forall$, *and* $\exists$ *enjoys the following properties in* $P(\Delta)$*:*

- $\Upsilon(\top) = \sup P(\Delta)$, *and* $\Upsilon(\bot) = \inf P(\Delta)$*;*
- $\Upsilon(X \sqcup Y) = \sup(\Upsilon(X), \Upsilon(Y))$, *and* $\Upsilon(X \sqcap Y) = \inf(\Upsilon(X), \Upsilon(Y))$*;*
- $\Upsilon(\forall R.\ X) = \sup\{S \in P(\Delta) : \forall s \in S \forall t \in \Delta((s,t) \in \Upsilon(R) \Rightarrow t \in \Upsilon(X))\}$,
  $\Upsilon(\exists R.\ X) = \sup\{S \in P(\Delta) : \forall s \in S \exists t \in \Delta((s,t) \in \Upsilon(R)\ \&\ t \in \Upsilon(X))\}$.

Conceptual interpretation is a formal context provided by an interpretation function.

Definition 4. Conceptual interpretation is a four-tuple (G, M, I, Υ), where (G, M, I) is a formal context and an interpretation function $\Upsilon = I_{CS} \cup I_{RS}$, where CS and RS are standard concept and role symbols, and

- $I_{CS} : CS \to \mathfrak{B}(G, M, I)$ maps concept symbols to formal concepts,
- $I_{RS} : RS \to 2^{(G \times G) \cup (M \times M)}$ maps role symbols to binary relations.

A formal context (G, M, I) or conceptual interpretation (G, M, I, Υ) is said to be homogeneous (symmetric) if $G = M$ (respectively, the binary relation I is symmetric).

Semantics of the $\mathcal{ALC}$ positive concept constructs $\top$, $\bot$, $\sqcup$, $\sqcap$, $\forall$, and $\exists$, as well as semantics of the negative construct $\neg$, are defined in [12] as follows.

Definition 5. Let (G, M, I, Υ) be a conceptual interpretation, $\mathbb{K}$ be a formal context (G, M, I), and $\mathfrak{B} = \mathfrak{B}(\mathbb{K})$ be the concept lattice of $\mathbb{K}$. The interpretation function Υ can be extended to all role terms in a terminological interpretation $((G \cup M), \Upsilon)$ in the standard manner so that $\Upsilon(R)$ is a binary relation on $(G \cup M)$ for every role term R. The interpretation function Υ can be extended to all positive $\mathcal{ALC}$ concept terms as follows.

- $\Upsilon(\top) = \sup \mathfrak{B}$ and $\Upsilon(\bot) = \inf \mathfrak{B}$;
- $\Upsilon(X \sqcup Y) = \sup(\Upsilon(X), \Upsilon(Y))$, and $\Upsilon(X \sqcap Y) = \inf(\Upsilon(X), \Upsilon(Y))$;
- Let $\Upsilon(X) = (Ex', In') \in \mathfrak{B}$. Then

– $\Upsilon(\forall R.\ X) = \sup_K\{(Ex, In) \in \mathfrak{B} :$
$\forall o \in Ex\ \forall a \in In\ \forall o' \in G\ \exists a' \in M$
$((o, o') \in \Upsilon(R) \Rightarrow o' \in Ex',\ (a, a') \in \Upsilon(R), \text{ and } a' \in In')\}$,

– $I(\exists R.\ X) = \sup_K\{(Ex, In) \in \mathfrak{B} :$
$\forall o \in Ex\ \forall a \in In\ \exists o' \in G\ \forall a' \in M$
$((a, a') \in \Upsilon(R) \Rightarrow (o, o') \in \Upsilon(R),\ o' \in Ex', \text{ and } a' \in In')\}$.

In addition, if $\mathbb{K}$ is a symmetric context and $\Upsilon(X) = (Ex, In) \in \mathfrak{B}$, then $\Upsilon(\neg X)$ can be defined as (In, Ex).

The following proposition [12] states that for any conceptual interpretation every positive $\mathcal{ALC}$ concept term is an element of a concept lattice; in addition, if an interpretation is symmetric, this fact holds for all $\mathcal{ALC}$ concept terms.

Proposition 2.

1. *For any conceptual interpretation (G, M, I, Υ), for every positive $\mathcal{ALC}$ concept term X, semantics $\Upsilon(X)$ is an element of $\mathfrak{B}(G, M, I)$.*
2. *For any symmetric conceptual interpretation (D, D, I, Υ), for every $\mathcal{ALC}$ concept term X, semantics $\Upsilon(X)$ is an element of $\mathfrak{B}(D, D, I)$.*

(Due to the lack of space, examples illustrating Proposition 2 and Proposition 3 below will be given in the full paper.)

Let (Δ, Υ) be a terminological interpretation. It is well-known that the powerset lattice $P(\Delta) = (2^\Delta, \subseteq, \varnothing, \Delta, \cup, \cap)$ is isomorphic to the concept lattice of a homogeneous formal context $K_\Delta^{\neq} = (\Delta, \Delta, \neq)$. A particular isomorphism is a function $\iota : 2^\Delta \to \mathfrak{B}(K_\Delta^{\neq})$ that maps every subset $S \subseteq \Delta$ to a formal concept $(S, \overline{S})$. This isomorphism defines conceptual interpretation $(\mathfrak{B}(\Delta, \Delta, \neq), \iota I)$, where (ιI) equals Υ on all object symbols and on all role symbols, and on concept symbols it is 'induced' by ι: $(\iota I)(p) = (\Upsilon(p), \overline{\Upsilon(p)})$ for every concept symbol p. The following proposition (also borrowed from [12]) demonstrates that semantics of $\mathcal{ALC}$ in terminological interpretation (Δ, Υ) and in conceptual interpretation $(\mathfrak{B}(\Delta, \Delta, \neq), \iota I)$ are closely connected. The following proposition is proved in [12] by induction on the structure of concept terms.

Proposition 3. *For every $\mathcal{ALC}$ concept term Z and every terminological interpretation (Δ, Υ), the following equality holds: $(\iota I)(Z) = \iota(\Upsilon(Z))$.*

Informally speaking, the above proposition states that semantics of $\mathcal{ALC}$ on concept lattices, defined in Definition 5, is compatible with the standard Kripke set-theoretic semantics of $\mathcal{ALC}$. Due to this interpretation of the proposition, we would like to refer to the proposition as the compatibility property, and consider it as a strong evidence for naturalness and soundness of our definition.

3. Ways to build a symmetric context

The above Proposition 2 leads to the following idea: to define the semantics of $\mathcal{ALC}$ with values in a concept lattice by isomorphic embedding of the background context into a symmetric one in such a way that for the positive fragment of $\mathcal{ALC}$ the original semantics and the induced semantics equal each other. Below we examine some opportunities to *symmetrize* a given context (i.e. to build a symmetric context from an arbitrarily given background context) by set-theoretic and algebraic manipulations with the binary relation of the context. Without loss of generality, we may assume that the background context is reduced [5] and has disjoint sets of objects and attributes.

Let $\mathbb{K} := (G, M, I)$ be a reduced context where $G \cap M = \varnothing$ and $\mathbb{K}^d = (M, G, I^d)$ be its dual context. Let us also use the following notation for binary relations (on M and/or G):

- $\varnothing$ is an empty binary relation,
- $\times$ is a total binary relation,
- E is an identity binary relation,
- E^c is a complement for E.

We would like to combine the cross-tables of $\mathbb{K}$ and the dual context $\mathbb{K}^d$ into the symmetric one in the following way:

$$\begin{array}{c|cc} & G & M \\ \hline G & ? & I \\ M & I^d & ? \end{array}$$

Let us represent the above cross-table in a shorter form as

$$\begin{array}{c|c} ? & \mathbb{K} \\ \hline \mathbb{K}^d & ? \end{array}$$

and denote the corresponding symmetric context by $\mathbb{K}_\circ := (G_\circ, M_\circ, I_\circ)$. Recall that $\mathfrak{B}(G, M, I)$ is the concept lattice of the context $\mathbb{K}$, $\mathfrak{B}(G_\circ, M_\circ, I_\circ)$ is the concept lattice of the context $\mathbb{K}_\circ$. Let us use the standard notation '$'$' for derivatives in the background context $\mathbb{K}$ and, for distinction, the notation '$^\circ$' for derivatives in the symmetric context.

We are going to fill free quadrants (i.e. with question marks) with different combinations of $\varnothing$, $\times$, E and E^c. In total, there are $2^4 = 16$ combinations. Below we study 9 of these 16 combinations.

Case 1 ($\varnothing$,$\varnothing$). Let us consider the case

$\varnothing$	$\mathbb{K}$
$\mathbb{K}^d$	$\varnothing$

It is the disjoint union of $\mathbb{K}$ and $\mathbb{K}^d$. The concept lattice $\mathfrak{B}(\mathbb{K}_\circ) = \mathfrak{B}(\mathbb{K}\dot{\cup}\mathbb{K}^d)$ is a *horizontal sum* [5], i.e. the union of two sublattices $\mathfrak{B}(\mathbb{K})$ and $\mathfrak{B}(\mathbb{K}^d)$ such that $\mathfrak{B}(\mathbb{K}) \cap \mathfrak{B}(\mathbb{K}^d) = \{\perp, \top\}$.

Case 2 ($\times$,$\varnothing$). Let us consider the case

$\times$	$\mathbb{K}$
$\mathbb{K}^d$	$\varnothing$

The concept lattice of this context is isomorphic to the *vertical sum* [5] of the concept lattices $\mathfrak{B}(\mathbb{K}^d)$ and $\mathfrak{B}(\mathbb{K})$ (where the concept lattice $\mathfrak{B}(\mathbb{K}^d)$ is over $\mathfrak{B}(\mathbb{K})$).

Case 3 ($\varnothing$,$\times$). This case is like the previous one, but we have to swap components of the vertical sum.

Case 4 ($\times$,$\times$). Let us consider the case

$\times$	$\mathbb{K}$
$\mathbb{K}^d$	$\times$

We have here the direct sum $\mathbb{K} + \mathbb{K}^d$ of the contexts $\mathbb{K}$ and $\mathbb{K}^d$ [5], and the concept lattice of the sum is isomorphic to the product of the concept lattices $\mathfrak{B}(\mathbb{K}) \times \mathfrak{B}(\mathbb{K}^d)$. A pair (A, B) is a concept of the direct sum $(\mathbb{K}+\mathbb{K}^d)$ iff $(A \cap G, B \cap M)$ is a concept of $\mathbb{K}$ and $(A \cap M, B \cap G)$ is a concept of $\mathbb{K}^d$. It implies that isomorphism is given by $(A, B) \mapsto ((A \cap G, B \cap M), (A \cap M, B \cap G))$.

Case 5 (E,E). Let us consider the case:

E	$\mathbb{K}$
$\mathbb{K}^d$	E

Let $(X, Y) \in \mathfrak{B}(\mathbb{K}_\circ)$ be a concept and let $X = A \dot{\cup} B$, where $A \subseteq G$, $B \subseteq M$. We have to consider the following 4 subcases:

1. $B = \varnothing$,
2. $A = \varnothing$,
3. $|B| = 1$ and $A \neq \varnothing$,
4. $|B| > 1$ and $A \neq \varnothing$.

Subcase 1. Let $X = A$. Then

$$X^\circ = A^\circ = A' \cup (\text{if } |A| = 1 \text{ then } \{a\} \text{ else } \varnothing) = Y,$$

i.e. if $|A| = 1$, then $(X, Y) = (\{a\}, \{a\} \cup A')$, and $(X, Y) = (A, A')$ otherwise.
Subcase 2. Let $X = B$. Then

$$X^\circ = B^\circ = B' \cup (\text{if } |B| = 1 \text{ then } \{b\} \text{ else } \varnothing) = Y,$$

i.e. if $|B| = 1$, then $(X, Y) = (\{b\}, \{b\} \cup B')$, and $(X, Y) = (B, B')$ otherwise.
Subcase 3. Let $X = A \cup \{b\}$. Then we have

$$\begin{aligned} X^\circ = & \\ &= A^\circ \cap \{b\}^\circ = (A' \cup (\text{if } |A| = 1 \text{ then } \{a\} \text{ else } \varnothing)) \cap (\{b\}' \cup \{b\}) = \\ &= (A' \cap \{b\}') \cup ((\text{if} \ldots) \cap \{b\}') \cup (A' \cap \{b\}) \cup ((\text{if} \ldots) \cap \{b\}) = \\ &= ((\text{if} \ldots) \cap \{b\}') \cup (A' \cap \{b\}) = \\ &= \text{if } (\{b\} \in A') \text{ then } (\text{if } |A| = 1 \text{ then } \{a, b\} \text{ else } \{b\}) \text{ else } \varnothing = \\ &= Y \end{aligned}$$

Hence, if $\{b\} \in A'$ and $|A| = 1$, then $(X, Y) = (\{a\} \cup \{b\}, \{a\} \cup \{b\})$, and if $\{b\} \in A'$ $|A| > 1$, then $(X, Y) = (A \cup \{b\}, \{b\})$.
Subcase 4. Let $|B| > 1$, then $X = A \cup B$. Then we have

$$\begin{aligned} X^\circ = & \\ &= A^\circ \cap B^\circ = (A' \cup (\text{if } |A| = 1 \text{ then } \{a\} \text{ else } \varnothing)) \cap B' = \\ &= (\text{if} \ldots) \cap B' = \text{if } (|A| = 1) \text{ then } (\{a\} \cap B') \text{ else } \varnothing = \\ &= \text{if } (|A| = 1) \text{ and } B \subseteq \{a\}' \text{ then } \{a\} \text{ else } \varnothing = \\ &= Y \end{aligned}$$

Hence, if $B \subseteq \{a\}'$ and $|A| = 1$, then $(X, Y) = (\{a\} \cup B, \{a\})$.

CASE 6 ($\varnothing$,E):

$$\begin{array}{c|c} \varnothing & \mathbb{K} \\ \hline \mathbb{K}^d & E \end{array}$$

This case is very similar to the previous one. Immediately we can present the result in each subcase of Case 5:
Subcase 1 ($B = \varnothing$). $(X, Y) = (A, A')$.
Subcase 2 ($A = \varnothing$). If $|B| = 1$ then $(X, Y) = (\{b\}, \{b\} \cup B')$ else $(X, Y) = (B, B')$.
Subcase 3 ($|B| = 1$). If $\{b\} \in A'$, we have $(X, Y) = (A \cup \{b\}, \{b\})$.
Subcase 4 ($|B| > 1$). All the concepts in this case will be either $\top$ or $\perp$.

CASE 7 (E,$\varnothing$). This case is similar to the previous one.

CASE 8 ($\times$,E). Now let us consider the case

$$\begin{array}{c|c} \times & \mathbb{K} \\ \hline \mathbb{K}^d & E \end{array}$$

Let us use subcases as in Case 5.
Subcase 1 ($B = \varnothing$). $(X, Y) = (A, G \cup A')$.
Subcase 2 ($A = \varnothing$). If $|B| = 1$, then $(X, Y) = (\{b\}, \{b\} \cup B')$, and $(X, Y) =$

(B, B') otherwise.
Subcase 3 ($|B| = 1$). If $\{b\} \in A'$, then $(X, Y) = (A \cup \{b\}, \{b\} \cup \{b\}')$, else $(X, Y) = (A \cup \{b\}, \{b\}')$.
Subcase 4 ($|B| > 1$). $(X, Y) = (A \cup B, B')$.

CASE 9 (E,$\times$). This case is similar to Case 8.

4. Conclusion

As we have already stated in the abstract, this paper is devoted to our progress in the studies of description logic with values in concept lattices. These studies are based on the approach presented in [12], the only definition (to the best of our knowledge) of a description logic with values in concept lattices.

The background motivation of this research is to expand web-search query languages from "googling" by key-words (i.e. computing the lower derivative) to more expressive language that admits the lower and upper derivatives in multiple contexts as well as Boolean combinations of queries. (Please refer to the Introduction for the example and discussion.)

Now we are ready to formulate some topics and problems that we consider natural and important for further research.

1. In Subsection 2.1, we represented the definition for modal logics with values in a given finite distributive lattice L; this definition is easy to extend to polymodal logics. In Subsection 2.2, we represented the definition for the description logic $\mathcal{ALC}$ (which can be considered as a polymodal version of $\mathbf{K}$) with values in concept lattices of symmetric contexts. Assume that $\mathbb{K}$ is a finite symmetric context; then $\mathfrak{B}(\mathbb{K})$ is a finite lattice, but it may not be a distributive lattice.
   Question: Assuming that $\mathfrak{B}(\mathbb{K})$ is a finite distributive lattice, is $\mathcal{ALC}$ with values in $\mathfrak{B}(\mathbb{K})$ a polymodal $\mathfrak{B}(\mathbb{K})$-**ML**?
2. In Subsection 2.2, we represented the definition for the description logic $\mathcal{ALC}$ with values in concept lattices of symmetric contexts and for the positive fragment of $\mathcal{ALC}$ with values in arbitrary concept lattices.
   Questions:
   (a) Is the positive fragment of $\mathcal{ALC}$ with values in concept lattices decidable/axiomatizable?
   (b) Is $\mathcal{ALC}$ with values in concept lattices of symmetric contexts decidable/axiomatizable?
3. In Section 3, we examine 9 of 16 variants of context symmetrization. Topics for further research are the following:

(a) to study the remaining seven cases of context symmetrization and isomorphic embedding with E^c in one or two free quadrants, and

(b) to examine under which embedding out of these sixteen the induced semantics of the positive fragment of $\mathcal{ALC}$ is equal to the original semantics.

4. One more topic: to investigate when an isomorphic embedding of a context to a symmetric context induces semantics for the positive fragment of $\mathcal{ALC}$ equal to the original semantics.

We also would like to study relations of the so-called *twist structures* (which are in use for the completeness of the algebraic semantics of Nelson logic) [10, 9] with $\mathcal{ALC}$ with values in concept lattices.

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