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Specification of monitor metrics for generating balanced numerical grids*

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Abstract. Formulas of monitor metrics are introduced for generating the vector field-aligned and/or adaptive grids. Some results of numerical experiments are demonstrated.

1. Formulation of the method

1.1. Mapping approach. Let an n-dimensional physical surface (in particular, a domain or a curve) be locally represented by parametrization

$$\boldsymbol{x}(\boldsymbol{s}) : S^n \to \mathbb{R}^{n+k}, \quad \boldsymbol{x} = (x^1, \dots, x^{n+k}), \quad \boldsymbol{s} = (s^1, \dots, s^n), \quad n \geq 1, \quad (1)$$

where S^n is an n-dimensional parametric domain (an interval when $n = 1$), while $\boldsymbol{x}(\boldsymbol{s})$ is a smooth vector-valued function of rank n at every point $\boldsymbol{s} \in S^n$. A physical geometry specified by (1) is denoted by S^{xn}. When $k = 0$, then S^{xn} is a domain $X^n \subset \mathbb{R}^n$ which itself may be considered as a parametric domain for X^n. The generation of a local numerical grid in S^{xn} is carried out by the mapping approach with the help of an intermediate nondegenerate smooth transformation

$$\boldsymbol{s}(\boldsymbol{\xi}) : \Xi^n \to S^n, \quad \boldsymbol{\xi} = (\xi^1, \dots, \xi^n), \quad (2)$$

between S^n and a suitable computational (logical) domain Ξ^n [1–4]. According to the approach, the grid nodes in S^{xn} are specified by mapping the nodes of a reference grid in Ξ^n with the transformation

$$\boldsymbol{x}[\boldsymbol{s}(\boldsymbol{\xi})] : \Xi^n \to S^{xn} \subset \mathbb{R}^{n+k}. \quad (3)$$

Depending on the form of S^{xn} and a numerical algorithm (finite differences, finite elements, finite volumes, spectral elements, etc.) applied for solving a physical problem, the computational domain Ξ^n and the cells of the reference grid may be rectangular or have an other shape. In particular, the reference grid may be unstructured [4].

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We designate by g_{ij}^{xv} the covariant metric elements of S^{xn} in the coordinates $v^1, \dots, v^n$. So, parameterizations (2) and (3) yield the corresponding formulas in the coordinates $s^1, \dots, s^n$ and $\xi^1, \dots, \xi^n$:

$$g_{ij}^{xs} = \boldsymbol{x}_{s^i} \cdot \boldsymbol{x}_{s^j}, \quad g_{ij}^{x\xi} = g_{mp}^{xs} \frac{\partial s^m}{\partial \xi^i} \frac{\partial s^p}{\partial \xi^j}, \quad i, j, m, p = 1, \dots, n.$$

Here and further the repeated indices mean that the summation is carried out over them.

1.2. Monitor metric. In addition to the computational domain Ξ^n and the reference grid, the present grid generation approach also assumes the introduction of a monitor manifold over S^{xn} by specifying a monitor metric at the points of S^{xn}. The monitor metric serves for controlling the properties of grids in the physical geometry S^{xn}. It is natural that the metric should be formulated through the quantities requiring grid adaptation: physical variables, geometric characteristics of S^{xn}, specified vector fields, etc. In addition, a mathematical formulation of the monitor metric should be simple and comprehensive so that an arbitrary grid could be realized by the approach considered.

The most general and simple formulation of the monitor metric in S^{xn}, denoted by $g_{ij}^{\boldsymbol{s}}$, $i, j = 1, \dots, n$ in the coordinates $s^1, \dots, s^n$, is given by the following formula:

$$g_{ij}^{\boldsymbol{s}} = z(\boldsymbol{s}) g_{ij}^{xs} + F_i^k(\boldsymbol{s}) F_j^k(\boldsymbol{s}), \quad i, j = 1, \dots, n, \quad k = 1, \dots, l, \tag{4}$$

where $z(\boldsymbol{s}) \geq 0$ is a weight function, g_{ij}^{xs} is the metric of S^{xn}, and $F_i^k(\boldsymbol{s})$, $i = 1, \dots, n$, are components of the vector $\boldsymbol{F}^k(\boldsymbol{s})$ [4].

The functions $z(\boldsymbol{s})$ and $F_i^k(\boldsymbol{s})$ in (4) are subject to the restriction: $\det(g_{ij}^{\boldsymbol{s}}) > 0$. In particular, $\det(g_{ij}^{\boldsymbol{s}}) > 0$ if $z(\boldsymbol{s}) > 0$.

1.3. The Dirichlet problem. A mathematical model for generating grids is formulated for an arbitrary physical geometry S^{xn}. Let us designate by $g_{ij}^{\boldsymbol{v}}$ the covariant elements in the coordinates $v^1, \dots, v^n$ of a monitor metric in S^{xn}. Then the intermediate transformation $\boldsymbol{s}(\boldsymbol{\xi})$ in (2) is defined as the inverse of the mapping

$$\boldsymbol{\xi}(\boldsymbol{s}) : S^n \to \Xi^n, \quad \boldsymbol{\xi}(\boldsymbol{s}) = [\xi^1(\boldsymbol{s}), \dots, \xi^n(\boldsymbol{s})],$$

which is subject to the Dirichlet problem:

$$\begin{aligned} &\frac{\partial}{\partial s^j}\Big(w(\boldsymbol{s}) g_{\boldsymbol{s}}^{jk} \frac{\partial \xi^i}{\partial s^k}\Big) = 0, \quad i, j, k = 1, \dots, n, \\ &\xi^i|_{\partial S^n} = \varphi^i(\boldsymbol{s}), \quad i = 1, \dots, n, \end{aligned} \tag{5}$$

where $g_{\boldsymbol{s}}^{jk}$ are the contravariant elements of the monitor metric in the coordinates $s^1, \dots, s^n$, $w(\boldsymbol{s}) > 0$ is a weight function aimed at controlling the effect of the metric at the points of S^{xn}, ∂S^n and $\partial \Xi^n$ are the boundaries of S^n and Ξ^n, respectively, while $\boldsymbol{\varphi}(\boldsymbol{s}) = [\varphi^1(\boldsymbol{s}), \dots, \varphi^n(\boldsymbol{s})]$ is a one-to-one continuous transformation between ∂S^n and $\partial \Xi^n$. The equations in (5) are referred to as the diffusive equations. The functions $\xi^1(\boldsymbol{s}), \dots, \xi^n(\boldsymbol{s})$ found as solutions of (5) specify a grid coordinate system in S^n and S^{xn}.

The diffusion equations in (5) are the Beltrami equations if $w(\boldsymbol{s}) = \sqrt{g^{\boldsymbol{s}}}$, $g^{\boldsymbol{s}} = \det(g_{ij}^{\boldsymbol{s}})$. Moreover, for $n \neq 2$ they are always the Beltrami equations, with respect to the metric

$$g_{ij} = (g^{\boldsymbol{s}})^{1/(2-n)}[w(\boldsymbol{s})]^{2/(n-2)} g_{ij}^{\boldsymbol{s}}, \quad i, j = 1, \dots, n \,, \tag{6}$$

regardless of the weight function $w(\boldsymbol{s})$.

Though the Beltrami equations are comprehensive [4], i.e., an arbitrary nondegenerate intermediate transformation (2) can be computed as the inverse of the solution to these equations, the form (5) of the diffusion equations with the weight function $w(\boldsymbol{s})$ appears to be more convenient, especially for $n = 2$, for realizing the necessary requirements for the grid properties in different zones of S^{xn}.

The system of equations in (5) is equivalent to that of the Euler–Lagrange equations of the following functional:

$$I[\boldsymbol{\xi}] = \int_{S^n} w(\boldsymbol{s}) g_{\boldsymbol{s}}^{jk} \frac{\partial \xi^i}{\partial s^k} \frac{\partial \xi^i}{\partial s^j} d\boldsymbol{s}, \quad i, j, k = 1, \dots, n. \tag{7}$$

The expression of functional (7) prompts one what form the monitor metric should take to provide the generation of a numerical grid with a required property. For this purpose, the metric is to have such a form that the integrand in (7)

$$\sigma(\boldsymbol{s}) = w(\boldsymbol{s}) g_{\boldsymbol{s}}^{jk} \frac{\partial \xi^i}{\partial s^j} \frac{\partial \xi^i}{\partial s^k}, \quad i, j, k = 1, \dots, n, \tag{8}$$

describes a measure of departure of the grid from the necessary grid at the point $\boldsymbol{s} \in S^n$. If such a metric is found, then it can be expected that the minimization of functional (7) will produce the grid with a required property.

2. Mathematical model for numerical implementations

2.1. Inverted equations. For generating grids, we have to interchange in the equations in the boundary value problem (5) their dependent and independent variables thus obtaining nonlinear elliptic equations with respect to the intermediate function $\boldsymbol{s}(\boldsymbol{\xi})$.

A system of the diffusion equations with respect to components of the intermediate transformation $\boldsymbol{s}(\boldsymbol{\xi})$

$$\frac{\partial}{\partial \xi^j}\left(w(\boldsymbol{\xi})g^{jk}\frac{\partial s^i}{\partial \xi^k}\right)=0, \quad i,j,k=1,\dots,n, \tag{9}$$

is linear and does not require any transformation, however the solution to this system can produce overlapping grids in concave domains, as is demonstrated by Figure 1.

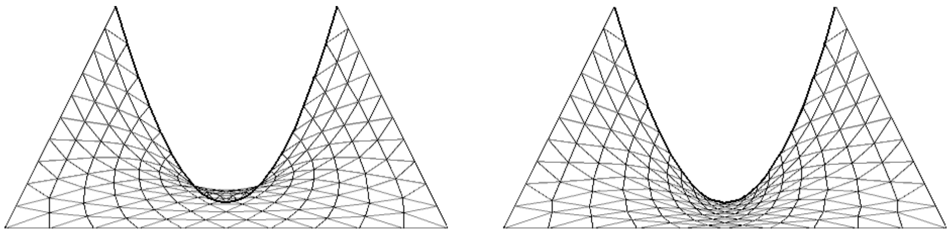

Figure 1. Grids in a concave domain generated by the solution to equations (9) (left) and by the solution to equations (5) (right); both with respect to the Euclidian metric

The boundary value problem (5) is transformed (see [4]) to the following inverted problem with respect to the dependent variables $s^i(\boldsymbol{\xi})$:

$$\begin{aligned} &w(\boldsymbol{s})g_{\boldsymbol{\xi}}^{km}\frac{\partial^2 s^i}{\partial \xi^k \partial \xi^m}=\frac{\partial}{\partial s^j}\left(w(\boldsymbol{s})g_{\boldsymbol{s}}^{ji}\right), \quad i,j,k,m=1,\dots,n, \\ &s^i|_{\partial \Xi^n}=\psi^i(\boldsymbol{\xi}), \quad i=1,\dots,n, \end{aligned} \tag{10}$$

where $g_{\boldsymbol{\xi}}^{ij}$ are the contravariant metric components of a monitor metric in the grid coordinates $\xi^1,\dots,\xi^n$, $\psi^i(\boldsymbol{\xi})$ is the ith component of the transformation inverse to $\boldsymbol{\varphi}(\boldsymbol{s})$. The numerical solution of (10) at the points of the reference grid in Ξ^n defines the nodes of the intermediate mesh in S^n. The image of these nodes by $\boldsymbol{x}(\boldsymbol{s})$ specifies the grid points in S^{xn}.

2.2. Specification of contravariant metric components. Note that both the functional of energy (7) and the inverted grid equations in (10) are formulated through the contravariant metric components $g_{\boldsymbol{s}}^{ij}$ and $g_{\boldsymbol{\xi}}^{ij}$ in the coordinates $s^1,\dots,s^n$ and $\xi^1,\dots,\xi^n$, respectively. Therefore instead of the covariant metric components $g_{ij}^{\boldsymbol{s}}$ one can originally, if convenient, formulate the contravariant components $g_{\boldsymbol{s}}^{ij}$ of the monitor metric, for example in the form (4), namely, as

$$g_{\boldsymbol{s}}^{ij}=\epsilon(\boldsymbol{s})g_{sx}^{ij}+B_k^iB_k^j, \quad i,j=1,\dots,n, \quad k=1,\dots,l, \tag{11}$$

where B_k^i, $i=1,\dots,n$, are components of the contravariant vector $\boldsymbol{B}_k=(B_k^1,\dots,B_k^n)$, $k=1,\dots,l$.

3. Specification of monitor metrics

3.1. Generation of vector field-aligned grids. A contravariant metric tensor in form (11) allows one to control the angle between a normal to a grid coordinate hypersurface and a specified vector field. As a tensor of the first rank in formula (11) one can either take the same or a transformed vector field. The generation of grids through such a metric is helpful for numerical solutions of problems with strong anisotropy, in particular, problems of plasma [5]. For example, the condition of orthogonality between the vector field $\boldsymbol{B} = (B^1, \ldots, B^n)$ specified at the points of a domain S^n and a normal to the coordinate hypersurface $\xi^1 = \text{const}$ can be described as the following equation of the quadratic form

$$(\boldsymbol{B} \cdot \operatorname{grad} \xi^1)^2 \equiv B^i B^j \frac{\partial \xi^1}{\partial s^i} \frac{\partial \xi^1}{\partial s^j} = 0, \quad i, j = 1, \ldots, n,$$

with a degenerate matrix $(B^i B^j)$. This quadratic form as measure of the grid departure from the field-alignment was used in [5] for generating nearly field-aligned grids in the domain S^n through the minimization of the functional

$$L = \int_{S^n} B^i B^j \frac{\partial \xi^1}{\partial s^i} \frac{\partial \xi^1}{\partial s^j} d\boldsymbol{s}, \quad i, j = 1, \ldots, n. \tag{12}$$

The integrand in the functional of energy (7) is formulated as sum of the quadratic forms

$$g_{\boldsymbol{s}}^{jk} \frac{\partial \xi^i}{\partial s^k} \frac{\partial \xi^i}{\partial s^j}, \quad i, j, k = 1, \ldots, n, \quad i \text{ fixed},$$

multiplied by $w(\boldsymbol{s})$, but contrary to (12) with a nondegenerate matrix $(g_{\boldsymbol{s}}^{jk})$. The condition of non-degeneracy is indispensable for obtaining unfolded grids through the minimization of functional (7). Therefore, in order to obtain grids which are both nearly field-aligned and unfolded, we have to slightly change the matrix $(B^i B^j)$ in functional (12) to make it nondegenerate. The matrix $(g_{\boldsymbol{s}}^{ij})$ whose elements are specified in form (11) is nondegenerate for an arbitrary $\epsilon(\boldsymbol{s}) > 0$, in addition, this matrix is close to the matrix $(B^i B^j)$ when both $\epsilon(\boldsymbol{s})$ and $\boldsymbol{B}_k$, $k = 2, \ldots, l$, are small and $\boldsymbol{B}_1 = \boldsymbol{B}$. Assume this matrix (with $g_{sx}^{ij} = \delta_j^i$) is a contravariant tensor of a monitor metric in the domain S^n. Then equations (10), aimed at the generation of grids provided that the angle between $\boldsymbol{B}$ and a normal to the coordinate hypersurface is close to $\pi/2$, have the form

$$w(\boldsymbol{s}) g_{\boldsymbol{\xi}}^{km} \frac{\partial^2 s^i}{\partial \xi^k \partial \xi^m} = \frac{\partial}{\partial s^j} \Big\{ w(\boldsymbol{s}) [(\epsilon(\boldsymbol{s}) \delta_j^i + B_a^i B_a^j] \Big\}, \tag{13}$$
$$i, j, k, m = 1, \ldots, n, \quad a = 1, \ldots, l.$$

3.2. A monitor metric for generating grids adapted to the gradient of a function. An efficient expression of the monitor metric for providing grid clustering in the zones of a large variation of the function $\boldsymbol{f}(\boldsymbol{s}) = (f^1(\boldsymbol{s}), \ldots, f^l(\boldsymbol{s}))$ was found in [6]. It has the following form

$$g^{\boldsymbol{s}}_{ij} = g^{xs}_{ij} + \frac{\partial \boldsymbol{f}}{\partial s^i} \cdot \frac{\partial \boldsymbol{f}}{\partial s^j}, \quad i, j = 1, \ldots, n. \tag{14}$$

A formula for the contravariant elements was described in [4]. In particular, when $g^{xs}_{ij} = \delta^i_j$, $\boldsymbol{f}(\boldsymbol{s})$ is a scalar-valued function $f(\boldsymbol{s})$, then

$$g^{ij}_{\boldsymbol{s}} = \delta^i_j - \frac{1}{1 + |\operatorname{grad} f|^2} \frac{\partial f}{\partial s^i} \frac{\partial f}{\partial s^j}, \quad i, j = 1, \ldots, n. \tag{15}$$

3.3. A monitor metric for generating grids adapted to the values of a function. For generating a numerical grid with the node clustering in the zones of large values of the function $v(\boldsymbol{s})$, the measure of departure from a required grid can be expressed in the form

$$\sigma(\boldsymbol{s}) = g[v](\boldsymbol{s}) g^{kl}_{sx} \frac{\partial \xi^i}{\partial s^k} \frac{\partial \xi^i}{\partial s^l}, \quad i, j, k, l = 1, \ldots, n, \tag{16}$$

where $g[v]$ is a positive operator such that $g[v](\boldsymbol{s})$ is large (small) where $v(\boldsymbol{s})$ is small (large). This measure for generating adaptive grids in domains was introduced in [7, 8]. Consequently, the contravariant elements of the monitor metric are the following:

$$g^{ij}(\boldsymbol{s}) = g[v](\boldsymbol{s}) g^{ij}_{sx}, \quad i, j = 1, \ldots, n. \tag{17}$$

This contravariant metric tensor can also be used for providing the node clustering in the zones of a large variation of the function $\boldsymbol{f}(\boldsymbol{s})$ by introducing for this purpose a function $v(\operatorname{grad} \boldsymbol{f})$ such that v is large where $|\operatorname{grad} \boldsymbol{f}|$ is large, and vice versa.

3.4. Monitor metrics for generating balanced grids. For computing balanced numerical grids that are field-aligned and adaptive to the values of one function and/or to variations of another function, a natural way for defining a monitor metric consists in combining the corresponding metrics, i.e., the contravariant metric elements are to have the form

$$g^{ij}(\boldsymbol{s}) = w_1(\boldsymbol{s}) g^{ij}_{\mathrm{al}} + w_2(\boldsymbol{s}) g^{ij}_{\mathrm{adg}} + w_3(\boldsymbol{s}) g^{ij}_{\mathrm{adv}}, \quad i, j = 1, \ldots, n, \tag{18}$$

where $w_i(\boldsymbol{s}) \geq 0$, $i = 1, 2, 3$, are weight functions specifying a contribution of the contravariant elements g^{ij}_{al}, g^{ij}_{adg}, and g^{ij}_{adv} formulated by (11), (15), and (17), respectively.

There may be other effective ways for combining the corresponding tensor components, in particular, for generating grids that are field-aligned and adaptive to the values of a function $f(\boldsymbol{s})$, adequate results demonstrates the formula

$$g^{ij}(\boldsymbol{s}) = g[f](\boldsymbol{s})g^{ij}_{\mathrm{al}}, \quad i,j = 1,\ldots,n. \tag{19}$$

4. Numerical experiments

4.1. Numerical grids aligned to vector-fields. For generating field-aligned numerical grids we used equations (10). These equations were solved in a two-dimensional physical domain X^2. We assumed $S^2 = X^2$ and Ξ^2 to be either as a square or a triangal with a uniform grid. The vector-field $\boldsymbol{B}$ was specified by

$$\boldsymbol{B} = \Big(-\frac{\partial g}{\partial s^2}, \frac{\partial g}{\partial s^1}\Big),$$

where $g(\boldsymbol{s})$ is a model function. The vector-field chosen is subject to the requirement for magnetic fields: $\operatorname{div} \boldsymbol{B} = 0$.

For generating a grid with a family of the grid coordinates aligned with the vector field $\boldsymbol{B}$, we assumed

$$g^{ij}_{\boldsymbol{s}} = \epsilon(\boldsymbol{s})\delta^i_j + B^i B^j, \quad i,j = 1,2.$$

The function $\epsilon(\boldsymbol{s})$ was formulated through the following boundary layer type functions [9]

$$\varphi(x,\delta) = \begin{cases} M\exp(-x/\delta), \\ M\delta^\alpha/(\delta + x)^\alpha, \quad \alpha > 0, \\ M(\delta + x)^\alpha, \quad 0 < \alpha < 1, \\ -M\ln(1 + x/\delta)/\ln\delta, \end{cases}$$

where $x \geq 0$, $0 < \delta \ll 1$, $M = \mathrm{const}$, assuming $\epsilon(\boldsymbol{s}) = \varphi(|\boldsymbol{B}(\boldsymbol{s})|^2, \delta)$. This function has small positive values when $|\boldsymbol{B}| \sim 1$, and close to 1 when $|\boldsymbol{B}| = 0$. Such functions help solutions to equations (10) switch from one mode to another.

Figures 2 and 3 demonstrate isocontours of the functions $g(\boldsymbol{s})$ and pictures of the corresponding grids. There were used the following expressions for $g(\boldsymbol{s})$ and $\epsilon(\boldsymbol{s})$: in Figure 2

$$g(\boldsymbol{s}) = v(s^2)(1 - v(s^2))[(s^1 - 0.5)^2 + 2(v(s^2) - 0.5)^2],$$
$$\varepsilon(\boldsymbol{s}) = \Big(\frac{0.3}{0.3 + |\boldsymbol{B}|^2}\Big)^5;$$

in Figure 3

$$g(\boldsymbol{s}) = v(s^2)(1 - v(s^2))[(s^1 - 0.5)^2 + 1.5(v(s^2) - 0.5)^2],$$
$$\varepsilon(\boldsymbol{s}) = 0.1\exp\Big(-\frac{|\boldsymbol{B}|}{0.07}\Big),$$

where

$$v(s^2) = 0.5\Big[1 + \tanh\Big(\frac{s^2 - 0.5}{0.2}\Big)\Big].$$

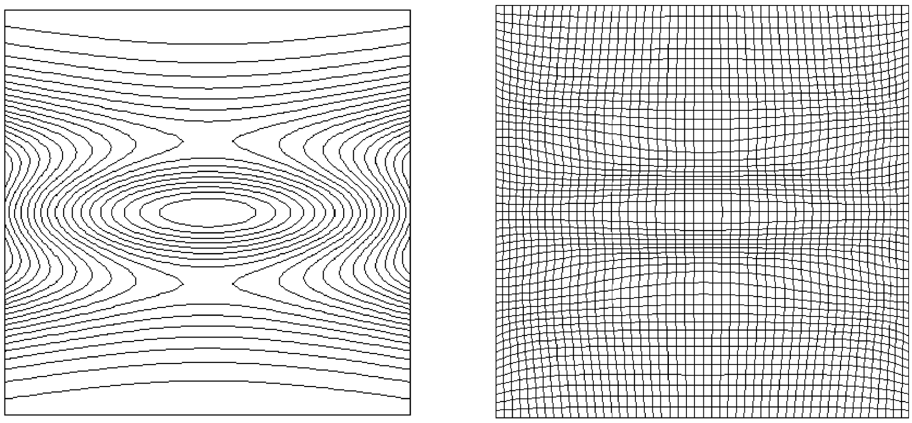

Figure 2

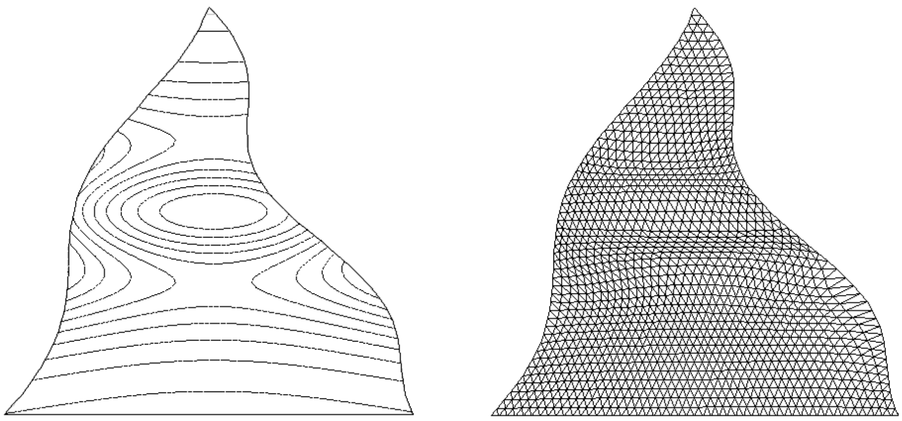

Figure 3

4.2. Pure adaptation to the gradient of a function $f(\boldsymbol{s})$. For generating grids with the node clustering in the zones of large values of the gradient of a function, the contravariant elements of the monitor metric were taken in form (15).

Figure 4 shows the grids adapted to the gradient of the following functions: to the left

$$f(\boldsymbol{s}) = 0.05 \tanh\Big(\frac{\varphi(\boldsymbol{s})}{0.05}\Big), \quad \varphi(\boldsymbol{s}) = 100(s^1 - 0.5)^2 + 16(s^2 - 0.5)^2 - 1;$$

to the right

$$f(\boldsymbol{s}) = 0.06 \tanh\Big(\frac{\varphi_1(\boldsymbol{s})}{0.05}\Big) + 0.08 \tanh\Big(\frac{\varphi_2(\boldsymbol{s})}{0.1}\Big),$$

$$\varphi_1(\boldsymbol{s}) = (s^1 - 0.5)^2 + (s^2 - 0.5)^2 - 0.5,$$

$$\varphi_2(\boldsymbol{s}) = s^2 - 0.5 - 0.8 \sin(6(s^1 + 0.3)).$$

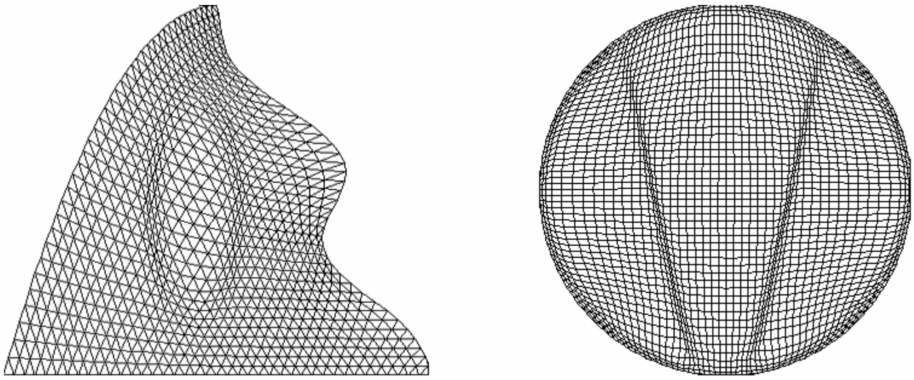

Figure 4

4.3. Pure adaptation to the values of a function $f(s)$. The contravariant elements of the monitor metric are specified as $g_{\boldsymbol{s}}^{ij} = \delta_j^i f(\boldsymbol{s})$.

Figure 5 demonstrates the grids adapted to the values of the following functions: to the left

$$f(\varphi) = \exp(-0.7\varphi),$$

$$\varphi(\boldsymbol{s}) = \exp\Big[-\frac{1}{0.35}\sin^2\Big(0.7\Big(\frac{(s^2-0.01)^2}{0.0625} + \frac{(s^1-0.5)^2}{0.0225} - 1\Big)\Big)\Big];$$

to the right

$$f(\varphi) = \Big(\frac{0.3}{0.3+\varphi}\Big)^3,$$

$$\varphi(\boldsymbol{s}) = \exp\Big(-\frac{\psi_1(\boldsymbol{s})}{0.005}\Big) + \exp\Big(-\frac{\psi_2(\boldsymbol{s})}{0.005}\Big) + 0.3\exp\Big(-\frac{\psi_3(\boldsymbol{s})^2}{0.002}\Big),$$

$$\psi_1(\boldsymbol{s}) = (s^1-0.35)^2 + (s^2-0.35)^2,$$

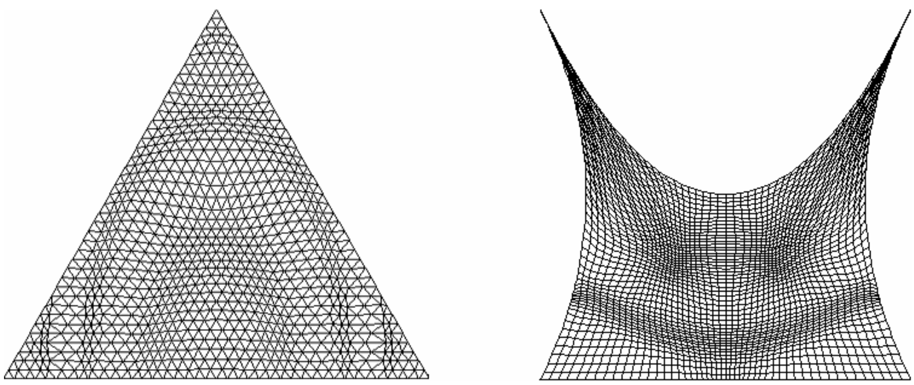

Figure 5

$$\psi_2(\boldsymbol{s}) = (s^1 - 0.65)^2 + (s^2 - 0.35)^2,$$
$$\psi_3(\boldsymbol{s}) = s^2 - (s^1 - 0.5)^2 - 0.1.$$

The right grid in Figure 5 was generated with the boundary adaptation with respect to the function $5\exp\Big(-\frac{\psi_3(\boldsymbol{s})^2}{0.01}\Big)$.

4.4. Balanced grids. For computing balanced numerical grids that are field-aligned and adaptive to the values of one function and to variations of another function, we used formula (18), written in the following form:

$$g^{ij}(\boldsymbol{s}) = (1-\alpha)g^{ij}_{\mathrm{al}} + \alpha\Big((1-\beta)g^{ij}_{\mathrm{adg}} + \beta g^{ij}_{\mathrm{adv}}\Big), \quad i,j = 1,\dots,n;$$
$$g^{ij}_{\mathrm{al}} = \delta^i_j \varepsilon(\boldsymbol{s}) + B^i B^j, \quad g^{ij}_{\mathrm{adv}} = \delta^i_j f_1(\varphi_1),$$
$$g^{ij}_{\mathrm{adg}} = \delta^i_j - \frac{1}{1 + |\operatorname{grad} f_2(\varphi_2)|^2}\frac{\partial f_2(\varphi_2)}{\partial s^i}\frac{\partial f_2(\varphi_2)}{\partial s^j}.$$

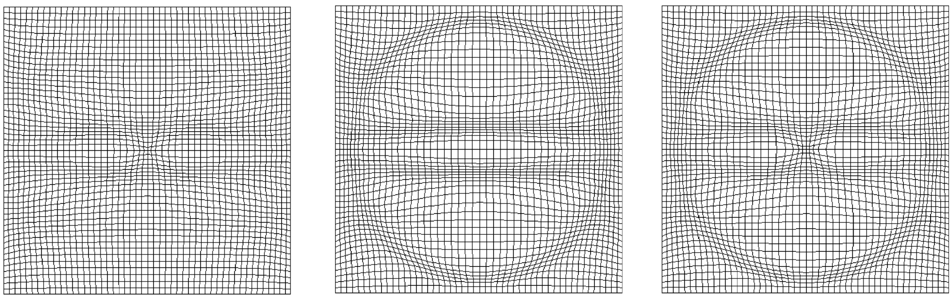

Figure 6

Some balanced grids are shown in Figure 6. The first picture of this figure demonstrates the grid aligned to the vector-field $\boldsymbol{B}$ and adapted to values of the function $f_1(\varphi_1)$. The second picture demonstrates the grid aligned to the same vector-field and adapted to the gradients of the function $f_2(\varphi_2)$. The third picture demonstrates the grid aligned to the same vector-field and adapted to values of one function and the gradients of the other. These grids were generated with the help of the following functions and parameters:

$$f_1(\varphi_1) = \Big(\frac{0.6}{0.6+\varphi_1}\Big)^3, \qquad \varphi_1(\boldsymbol{s}) = \Big(\frac{0.01}{0.01+R^2}\Big)^5,$$
$$f_2(\varphi_2) = 0.05\tanh\Big(\frac{\varphi_2}{0.03}\Big), \quad \varphi_2(\boldsymbol{s}) = R^2 - 0.2,$$
$$R^2 = (s^1 - 0.5)^2 + (s^2 - 0.5)^2,$$
$$g(\boldsymbol{s}) = v(s^2)(1 - v(s^2))[(s^1 - 0.5)^2 + 2(v(s^2) - 0.5)^2],$$

$$v(s^2) = 0.5\Big[1 + \tanh\Big(\frac{s^2 - 0.5}{0.2}\Big)\Big];$$

$$1)\ \ \alpha = \varepsilon(\boldsymbol{s}) = \Big(\frac{0.3}{0.3 + |\boldsymbol{B}|^2}\Big)^6, \quad \beta = 1,$$

$$2)\ \ \alpha = \varepsilon(\boldsymbol{s}) = \Big(\frac{0.3}{0.3 + |\boldsymbol{B}|^2}\Big)^8, \quad \beta = 0,$$

$$3)\ \ \alpha = \varepsilon(\boldsymbol{s}) = \Big(\frac{0.3}{0.3 + |\boldsymbol{B}|^2}\Big)^5, \quad \beta = \exp(-(f_1)^2/0.1).$$

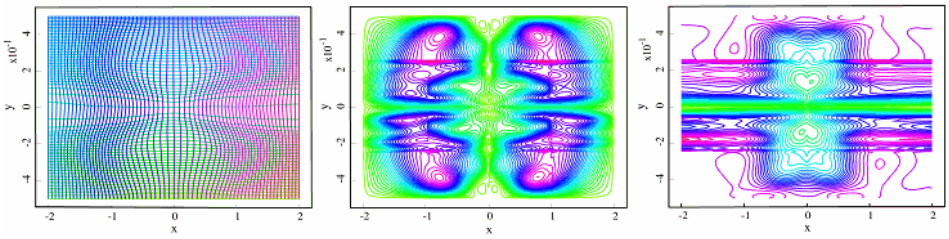

Figure 7

Figure 7 illustrates a balanced grid aligned to a magnetic field and adapted to the numerical error (the first picture), the alignment error (the second picture), and the scaled grid density (the third picture).

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